Calculus Notes
Grinshpan

POWER SERIES

A power series centered at x_0 is an infinite sum of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n + \ldots\ .$$

A common choice for the center x_0 is 0. The numbers $a_0, a_1, a_2, \ldots$ are the coefficients of the power series.

For any given value of x, the powers series becomes a regular infinite series of numbers, either convergent or divergent. For instance, if $x = x_0$ one always obtains a convergent sum: $a_0 + 0 + 0 + 0 + \ldots = a_0$.

EXAMPLE. The power series

$$1 + x + x^2 + \ldots + x^n + \ldots$$

is centered at $x_0 = 0$; its coefficients are $a_n = 1$, $n = 0, 1, 2, \ldots$. This is our old friend, the geometric series with ratio x. For every x with $|x| < 1$ the power series converges (to $\frac{1}{1-x}$) and for all other x it diverges.

THEOREM. The set of all values of x such that the power series converges is an interval centered at x_0, called the *interval of convergence*. The distance R from x_0 to either end of this interval is the *radius of convergence*. Three alternatives are possible:

$R = 0$, the power series converges for $x = x_0$ only,
$R > 0$, the power series converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$,
$R = \infty$, the power series converges for all x without restriction.

For every x with $|x - x_0| < R$, the power series converges *absolutely*. If R is finite and positive, the endpoints $x = x_0 \pm R$ must be considered separately: they may or may not be points of convergence. For x varying in the interval of convergence, the power series represents a function of x, namely, its sum.

To understand the logic behind this theorem consider the next example.

EXAMPLE. The power series

$$x + \tfrac{1}{2}\,x^2 + \tfrac{1}{3}\,x^3 + \ldots + \tfrac{1}{n}\,x^n + \ldots$$

is centered at $x_0 = 0$, its coefficients are $a_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$. Fix x and apply the Ratio test:

$$\frac{|x|^{n+1}}{n+1} : \frac{|x|^n}{n} = \frac{n}{n+1}\,|x| \to |x|, \quad n \to \infty.$$

Hence for every x with $|x| < 1$ the power series converges and, in fact, absolutely. For every x with $|x| > 1$ the power series diverges. For $x = \pm 1$ the Ratio test is inconclusive, these cases need separate consideration (endpoint analysis).

If $x = 1$, the power series becomes harmonic:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots .$$

Divergent. If $x = -1$, the resulting series

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

converges by the alternating series test.
It follows that the interval of convergence is $-1 \le x < 1$.
In this interval the sum of the power series can be shown to be $\ln(\frac{1}{1-x})$.

THEOREM. The radius of convergence of the power series can be found by either formula

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}, \qquad R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

provided that the limit in question exists.

EXAMPLE. The power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is centered at $x_0 = 0$; its coefficients are $a_n = \frac{1}{n!}$, $n = 0, 1, 2, \dots$. Observe that

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

Hence the radius of convergence is infinite. This means that the power series converges for all values of x. The function represented by this power series is e^x.