Properties of Riemann-integrable functions

Underlying properties of Darboux sums.
If \(m \leq f \leq M\), then \(m(b - a) \leq U(f, P) \leq U(f, P) \leq M(b - a)\)
If \(P \subset P'\), then \(U(f, P) \geq U(f, P')\) and \(L(f, P) \leq L(f, P')\)

\[U(f + g, P) \leq U(f, P) + U(g, P)\] and \(L(f + g, P) \geq L(f, P) + L(g, P)\)

If \(f_n \to f\), then \(U(f_n, P) \to U(f, P)\) and \(L(f_n, P) \to L(f, P)\)

Set additivity. Let \(c \in (a, b)\). Then \(f\) is Riemann-integrable on \([a, b]\) if and only if it is Riemann-integrable on \([a, c]\) and on \([c, b]\), in which case

\[
\int_a^c f = \int_a^b f + \int_c^b f.
\]

Proof. Every partition \(P\) of \([a, b]\) containing \(c\) is the union of \(P_1 = P \cap [a, c]\) and \(P_2 = P \cap [c, b]\). So if one side of the equality

\[U(f, P) - L(f, P) = U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2),\]

can be made arbitrarily small, the same is true of the other. Moreover,

\[
\int_a^b f = \inf_P U(f, P) = \inf_P (U(f, P_1) + U(f, P_2)) = \int_a^c f + \int_c^b f. \quad \Box
\]

Remark. For a fixed bounded \(f\) defined on \(\mathbb{R}\), the integral \(\mu([a, b]) = \int_a^b f\) provides us with a way to measure intervals and, perhaps, more general sets. The preceding property asserts that \(\mu\) is an additive set function: \(\mu([a, b]) = \mu([a, c]) + \mu([c, b])\).

Function additivity. If \(f, g\) are Riemann-integrable on \([a, b]\), then so is \(f + g\) and

\[
\int_a^b (f + g) = \int_a^b f + \int_a^b g.
\]

Proof. \(f + g\) is integrable on \([a, b]\), because for any partition \(P\)

\[U(f + g, P) - L(f + g, P) \leq U(f, P) - L(f, P) + U(g, P) - L(g, P),\]

and the right side can be made arbitrarily small. To prove additivity, choose \(P\) so that

\[U(f, P) - \frac{\varepsilon}{2} < \int_a^b f < L(f, P) + \frac{\varepsilon}{2} \quad \text{and} \quad U(g, P) - \frac{\varepsilon}{2} < \int_a^b g < L(g, P) + \frac{\varepsilon}{2}.
\]

Then

\[
\int_a^b (f + g) \leq U(f + g, P) \leq U(f, P) + U(g, P) < \int_a^b f + \int_a^b g + \varepsilon
\]

and

\[
\int_a^b (f + g) \geq L(f + g, P) \geq L(f, P) + L(g, P) > \int_a^b f + \int_a^b g - \varepsilon.
\]

So

\[
\int_a^b f + \int_a^b g - \varepsilon < \int_a^b (f + g) < \int_a^b f + \int_a^b g + \varepsilon \quad \text{for every} \quad \varepsilon > 0. \quad \Box
\]
**Homogeneity.** If $f$ is Riemann-integrable on $[a, b]$, then so is $cf$, $c \in \mathbb{R}$, and

$$\int_a^b cf = c \int_a^b f.$$  

**Proof.** For $c > 0$, $\inf_P U(cf, P) = c \inf_P U(f, P) = c \sup_P L(f, P) = \sup_P U(cf, P)$, and the assertion follows. For $c = -1$, $U(-f, P) = -L(f, P)$ and $L(-f, P) = -U(f, P)$. □

**Remark.** The class of Riemann-integrable functions on $[a, b]$ is a (real) vector space, as it is closed under addition and scaling.

**Monotonicity.** If $f$ and $g$ are Riemann-integrable on $[a, b]$ and if $f \leq g$, then

$$\int_a^b f \leq \int_a^b g.$$  

**Proof.** $\int_a^b g - \int_a^b f = \int_a^b (g - f) = \inf_P U(g - f, P) \geq 0$. □

**Remark.** See also Problem 1 of Homework 8.

**Triangle inequality:** If $f$ is Riemann-integrable on $[a, b]$ then so is $|f|$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$  

**Proof:** Since $||x| - |y|| \leq |x - y|$, $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$, which implies that $|f|$ is integrable. By monotonicity, $-|f| \leq f \leq |f|$ yields the triangle inequality. □

**Uniform limits.** If $f_n$ are Riemann-integrable and $f_n \rightarrow f$ on $[a, b]$, then $f$ is Riemann-integrable as well and

$$\int_a^b f_n \rightarrow \int_a^b f.$$  

**Proof.** Given $\varepsilon > 0$, select $n$ so that $|f - f_n| < \varepsilon/2(b - a)$ on $[a, b]$. Then

$$U(f, P) - L(f, P) \leq U(f_n, P) - L(f_n, P) + U(f - f_n, P) - L(f - f_n, P) \leq U(f_n, P) - L(f_n, P) + \varepsilon/2.$$  

Now select $P$ so that $U(f_n, P) - L(f_n, P) < \varepsilon/2$.

To prove the convergence of integrals, use that

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \int_a^b |f - f_n|$$

and that the right-hand side tends to zero as $n \rightarrow \infty$. □