Notes on real and complex differentiability

Real differentiability in one variable

A real function $f(x)$ is said to be differentiable at x_0, an interior point of its domain, if the ratio of $\Delta f = f(x) - f(x_0)$ to $\Delta x = x - x_0$ has a limit as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = a.$$

The limit value a is denoted by $f'(x_0)$. The differentiability property can be rephrased as a statement on the existence of a linear part: there exists a real number a such that

$$\Delta f = a\,\Delta x + o(\Delta x), \quad \Delta x \to 0,$$

where $o(\Delta x)$ designates a quantity that tends to 0 faster than Δx:

$$\lim_{\Delta x \to 0} \frac{o(\Delta x)}{\Delta x} = 0.$$

Example. $f(x) = x^2$ is differentiable at $x_0 = 1$: $x^2 - 1 = \underbrace{2(x-1)}_{f'(1)\,\Delta x} + \underbrace{(x-1)^2}_{o(\Delta x)}$, as $x \to 1$.

Complex differentiability

A complex function $f(z)$ of $z = x + iy$ is said to be differentiable at z_0, an interior point of its domain, if the ratio of $\Delta f = f(z) - f(z_0)$ to $\Delta z = z - z_0$ has a limit as $\Delta z \to 0$:

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = c.$$

Equivalently, there exists a complex number c such that

$$\boxed{\Delta f = c\,\Delta z + o(\Delta z), \quad \Delta z \to 0}.$$

The value of c is denoted by $f'(z_0)$.

Example. $f(z) = z^2$ is differentiable at $z_0 = 1$: $z^2 - 1 = \underbrace{2(z-1)}_{f'(1)\,\Delta z} + \underbrace{(z-1)^2}_{o(\Delta z)}$, as $z \to 1$.

The analogy between the two definitions can be misleading: the assumption of complex differentiability, as we shall see, imposes some very strong conditions on the function.
In particular, the value of c must be independent of the path that Δz takes to 0.

Cauchy–Riemann equations

Suppose that $f'(z_0)$ exists.

Let $\Delta z = \Delta x + i\Delta y \to 0$ through real values: $\Delta z = \Delta x$, $y = y_0$. Then

$$\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x\to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x} + i\,\frac{\Delta v}{\Delta x} = u_x(x_0,y_0) + iv_x(x_0,y_0) = f_x(z_0).$$

Let now $\Delta z \to 0$ through imaginary values: $\Delta z = i\Delta y$, $x = x_0$. Then

$$\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta y\to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = \lim_{\Delta y\to 0} -i\,\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta y} = v_y(x_0,y_0) - iu_y(x_0,y_0) = -if_y(z_0).$$

Thus

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0).$$

It follows that the real and imaginary parts of f must have partial derivatives at (x_0,y_0) and that the partial derivatives must satisfy the conditions

$$u_x(x_0,y_0) = v_y(x_0,y_0), \quad u_y(x_0,y_0) = -v_x(x_0,y_0).$$

These are the Cauchy–Riemann equations.

The Cauchy–Riemann equations may be written as a single complex equation

$$f_x(z_0) + if_y(z_0) = 0.$$

Real differentiability in two variables

A function $f(x,y)$ of two real valiables x and y is said to be differentiable at (x_0,y_0), an interior point of its domain, if there exist constants a, b such that

$$\Delta f \;=\; a\Delta x + b\Delta y + o\Big(\sqrt{(\Delta x)^2 + (\Delta y)^2}\Big), \quad (\Delta x,\Delta y) \to (0,0),$$

in which case $a = f_x(x_0,y_0)$ and $b = f_y(x_0,y_0)$ are the partial derivatives.

Viewing $f(x,y)$ as a function of $z = x+iy$, we can rewrite the preceding expression:

$$\begin{aligned}\Delta f &= \tfrac{1}{2}(a-ib)(\Delta x + i\Delta y) + \tfrac{1}{2}(a+ib)(\Delta x - i\Delta y) + o\Big(\sqrt{(\Delta x)^2 + (\Delta y)^2}\Big)\\ &= c_1\Delta z + c_2\Delta\overline{z} + o(|\Delta z|).\end{aligned}$$

Thus a complex mapping $f(x+iy)$ is real differentiable at (x_0,y_0) if there exist complex numbers c_1, c_2 such that

$$\boxed{\Delta f \;=\; c_1\Delta z + c_2\Delta\overline{z} + o(\Delta z), \quad \Delta z \to 0}.$$

Note that $c_1 = \dfrac{f_x(x_0,y_0) - if_y(x_0,y_0)}{2}$ and $c_2 = \dfrac{f_x(x_0,y_0) + if_y(x_0,y_0)}{2}$.

Differentiability criterion

Comparing the boxed conditions we arrive at the following.

A complex function $f(x+iy) = u + iv$ is differentiable at $z_0 = x_0 + iy_0$ in the complex sense if and only if it is differentiable at (x_0, y_0) in the real sense and $c_2 = 0$, i.e., the Cauchy–Riemann condition $f_x + if_y = 0$ holds at z_0, in which case $f'(z_0) = c_1$.

The circle of limit values

Suppose that a complex mapping $f(z) = f(x+iy)$ is differentiable at $z_0 = x_0 + iy_0$ in the real sense, as a function of x and y. Let us examine the possible limit values of the difference quotient $\Delta f/\Delta z$, as $z \to z_0$. Since

$$\Delta f \;=\; c_1\Delta z + c_2\Delta\overline{z} + o(\Delta z), \quad \Delta z \to 0,$$

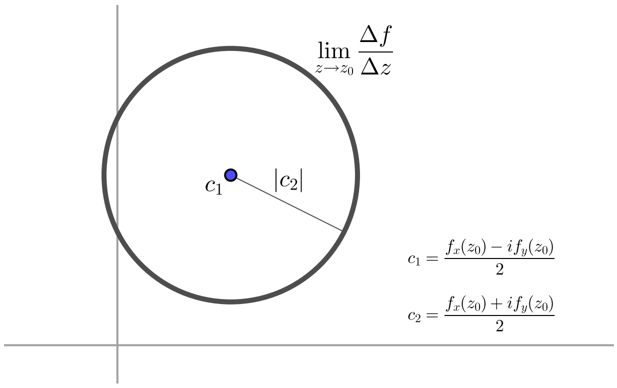
we have

$$\frac{\Delta f}{\Delta z} \;=\; c_1 + c_2\,\frac{\Delta\overline{z}}{\Delta z} + o(1), \quad \Delta z \to 0.$$

Let $\Delta z = z - z_0 \to 0$ radially, along the ray of argument θ. We may write $\Delta z = r\lambda$, where $\lambda = \cos\theta + i\sin\theta$ is a fixed point on the unit circle and $r = |\Delta z| \to 0$. Then

$$\lim_{r\to 0}\frac{\Delta f}{\Delta z} \;=\; c_1 + c_2\,\frac{\overline{\lambda}}{\lambda} \;=\; c_1 + c_2\overline{\lambda}^2.$$

Therefore the locus of radial limit values of $\Delta f/\Delta z$ is the circle $|z - c_1| = |c_2|$.
In fact, every possible limit value of the difference quotient must necessarily belong to this circle. The circle degenerates into a point if and only if $f'(z_0)$ exists.



Example. The linear function $f(z) = 2z + 0.1\,\overline{z}$ is differentiable in the real sense.
Fix any z_0 and let $\Delta z = |\Delta z|\lambda$, where $\lambda = \cos\theta + i\sin\theta$. Then

$$\frac{\Delta f}{\Delta z} \;=\; \frac{2\Delta z + 0.1\,\Delta\overline{z}}{\Delta z} \;=\; 2 + 0.1\,\overline{\lambda}^2.$$

Observe that the difference quotient is constant along each ray $z = z_0 + r\lambda$ and that the locus of directional derivatives of f at z_0 is the circle $|z - 2| = 0.1$ (doubly traced).
The function f is not differentiable in the complex sense, but it "comes close" to being complex differentiable, because the circle is small.

Necessary conditions for complex differentiability

The Cauchy–Riemann equations are necessary for complex differentiability.

However, these equations are not sufficient, because the existence of first partial derivatives does not guarantee real differentiability, let alone continuity.

Example. The function $f(x,y) = xy/(x^2+y^2)$, with specification $f(0,0) = 0$, has first partial derivatives at every point. As $f(x,0) = f(0,y) = 0$, we have $f_x(0,0) = f_y(0,0) = 0$. But $f(x,y)$ is not continuous at the origin, and so is not differentiable there.

Example [Looman, 1923]. The function $f(z) = \exp(-1/z^4)$, with specification $f(0) = 0$, satisfies the Cauchy–Riemann equations everywhere, but is not continuous, and not differentiable, at the origin: $\lim_{s\to 0} f(s+is) = \lim_{x\to 0^-} e^{-1/x} = \infty$.

Example [Menshov, 1936]. The function $f(z) = z^5/|z|^4$, with specification $f(0) = 0$, is continuous everywhere and satisfies the Cauchy–Riemann equations at the origin, but fails to be differentiable at the origin: $f(z)/z = (z/|z|)^4$ does not have a limit as $z \to 0$.

Sufficient conditions for complex differentiability

Recall that a real function that has first partial derivatives is differentiable at every point where those partial derivatives are continuous.

Let the first partial derivatives of u and v exist in an open neighborhood of z_0. If these partial derivatives are continuous at z_0 and satisfy the Cauchy–Riemann equations at z_0, then $f = u + iv$ is differentiable at z_0 in the complex sense.

Holomorphic functions

A complex-valued function that is defined in an open subset G of the complex plane and is differentiable at every point of G is said to be holomorphic (analytic, regular) in G.

Our discussion shows that the real and imaginary parts of a holomorphic function have first partial derivatives and satisfy the Cauchy–Riemann equations.

Conversely, if the real and imaginary parts of a complex function have continuous first partial derivatives and obey the Cauchy–Riemann equations, the function is holomorphic.

With an additional provision of continuity, the conditions of Cauchy–Riemann and holomorphy become equivalent.

Looman–Menshov theorem (1923, 1936). Let a function $f = u + iv$ be defined and continuous in an open set G. If u and v have first partial derivatives and satisfy the Cauchy–Riemann equations in G, then f is holomorphic in G.