Approximation of Area

A  Estimate the area below $y = x^2$ and over $[-3, 5]$ using 4 rectangles of equal width. Consider three cases: left endpoints, right endpoints, and midpoints.

B  Form the right endpoint sum $R_n$ using $n$ rectangles of equal width.

C  Find a closed-form expression for $R_n$.

D  Find the exact area under the graph by letting $n \to \infty$.

Solution

A  The three approximations are:

$$L_4 = 2 (9 + 1 + 1 + 9) = 40$$
$$R_4 = 2 (1 + 1 + 9 + 25) = 72$$
$$M_4 = 2 (4 + 0 + 4 + 16) = 48$$

B  The length of the interval $[-3, 5]$ is 8 units. Subdivide $[-3, 5]$ into $n$ equal subintervals of length $h = 8/n$ each. The corresponding right endpoint sum for $f(x) = x^2$ is then:

$$R_n = h \left( (-3 + h)^2 + (-3 + 2h)^2 + \ldots + (-3 + nh)^2 \right)$$

$$= h \sum_{k=1}^{n} (-3 + kh)^2$$

$$= \frac{8}{n} \sum_{k=1}^{n} \left( \frac{8}{n} k - 3 \right)^2.$$  

C  Breaking the sum in three, factoring out constants, and using summation formulas, we have:

$$\sum_{k=1}^{n} \left( \frac{8k}{n} - 3 \right)^2 = \sum_{k=1}^{n} \left( \frac{64}{n^2} k^2 - \frac{48}{n} k + 9 \right)$$

$$= \frac{64}{n^2} \sum_{k=1}^{n} k^2 - \frac{48}{n} \sum_{k=1}^{n} k + \sum_{k=1}^{n} 9$$

$$= \frac{64}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{48}{n} \frac{n(n+1)}{2} + 9n$$

$$= \frac{32}{3} \frac{(n+1)(2n+1)}{n} - 24(n+1) + 9n$$

$$= \frac{3}{3} n + 8 + \frac{32}{3n}.$$
Consequently,

\[ R_n = \frac{8}{n} \left( \frac{19}{3} n + 8 + \frac{32}{3n} \right) \]

\[ = \frac{152}{3} + \frac{64}{n} + \frac{256}{3n^2}. \]

This is the total area covered by \( n \) right rectangles.

D Letting \( n \to \infty \), we find the exact area under the graph,

\[
\int_{-3}^{5} x^2 \, dx = \lim_{n \to \infty} R_n = \frac{152}{3} = 50 \frac{2}{3}.
\]