TAYLOR POLYNOMIALS

THE REMAINDER TERM

Let \( f \) be an \( n \)-times differentiable function defined on an interval, and let \( x_0 \) and \( x \) be points in its domain. Consider the \( n \)-th Taylor polynomial of \( f \) centered at \( x_0 \) and evaluated at \( x \),

\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k.
\]

Our goal is to get a hold of the remainder term \( f(x) - T_n(x) \).

The formulas stated below are valid under the assumption that the \( n \)-th derivative \( f^{(n)} \) is continuously differentiable. Actually, they hold under a slightly weaker but more technical assumption that \( f^{(n)} \) is continuous on the closed interval with endpoints \( x_0 \) and \( x \) and is differentiable in the open interval with those endpoints.

AN INTEGRAL FORM OF THE REMAINDER

\[
f(x) - T_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(s)(x-s)^n ds.
\]

THE LAGRANGE FORM OF THE REMAINDER

\[
f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1},
\]

for some \( c \) in the open interval with endpoints \( x_0 \) and \( x \).

Both formulas are tied to the Mean-Value Theorem and may be used for error estimates.

EXAMPLE The \( n \)-th Taylor polynomial of \( f(x) = e^x \) at \( x_0 = 0 \) is

\[
T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots + \frac{1}{n!} x^n.
\]

Set \( x = 1 \). Then, by the Lagrange remainder formula,

\[
e^1 - T_n(1) = \frac{e^c}{(n+1)!},
\]

where \( c = c(n) \) satisfies \( 0 < c < 1 \). The sums \( T_n(1) \) are monotonically increasing with \( n \).

Using that \( 1 < e^c < 3 \), we obtain simple bounds

\[
\frac{1}{(n+1)!} < e - T_n(1) < \frac{3}{(n+1)!}.
\]

In particular, the limit of \( T_n(1) \), as \( n \to \infty \), is \( e \). It is easy to check that \( 3/13! < 5 \times 10^{-10} \), and so \( T_{12}(1) = \sum_{k=0}^{12} \frac{1}{k!} = 2.718281828\ldots \) has 10 significant digits of \( e \).