Mapping $z \mapsto \sin z$

The complex sine function

$$\sin z \ = \ \frac{e^{iz} - e^{-iz}}{2i}$$

reduces to the usual $\sin x$ when its argument $z = x$ is real.

It may be thought of as a composition of $-ie^{iz}$ with the Joukovsky function

$$J(z) \ = \ \frac{z + z^{-1}}{2}.$$

Let $\sin z = u + iv$ be restricted to the half-infinite vertical strip

$$0 \le x \le \frac{\pi}{2}, \quad y \ge 0.$$

We will verify that the one-to-one image of the strip is the first quadrant.

Observe first that

$$u \ = \ \mathrm{Re}\,\frac{e^{ix-y} - e^{-ix+y}}{2i} \ = \ \frac{e^{-y}\sin x + e^{y}\sin x}{2} \ = \ \sin x \cosh y \ \ge \ 0$$

and

$$v \ = \ \mathrm{Im}\,\frac{e^{ix-y} - e^{-ix+y}}{2i} \ = \ \frac{-e^{-y}\cos x + e^{y}\cos x}{2} \ = \ \cos x \sinh y \ \ge \ 0.$$

The boundary correspondence can now be tracked.

As $z = x$ moves right along the real line segment $[0,\ \frac{\pi}{2}]$, the image point $w = \sin x$ traces the real line segment $[0,\ 1]$ in the same direction.

As $z = \frac{\pi}{2} + iy$ moves up along the extended vertical ray $[\frac{\pi}{2} + i0,\ \frac{\pi}{2} + i\infty]$, the image point $w = \cosh y$ travels the extended real ray $[1,\ \infty]$ to the right.

As $z = iy$ comes down along the positive imaginary ray $[i0,\ i\infty]$, its image $w = i\sinh y$ traces the positive imaginary ray $[i0,\ i\infty]$ in the same direction.

Next, let us examine the image of the Cartesian grid lines.

For a fixed value of $x_0 \in (0,\ \frac{\pi}{2})$, the extended vertical ray $[x_0 + i0,\ x_0 + i\infty]$ is mapped onto the hyperbolic arc

$$\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = 1, \quad u > 0,\ v \ge 0.$$

Since $\sin^2 x_0 + \cos^2 x_0 = 1$, the foci of the hyperbola are at ± 1.

For a fixed value of $y_0 > 0$, the horizontal line segment $[0 + iy_0,\ \frac{\pi}{2} + iy_0]$ is mapped onto the elliptic arc

$$\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1, \quad u, v \geq 0.$$

Since $\cosh^2 y_0 - \sinh^2 y_0 = 1$, the foci of the ellipse are at ± 1.

Thus the images of the vertical grid lines in the interior of the strip are the hyperbolic arcs

$$\frac{u^2}{r^2} - \frac{v^2}{1 - r^2} = 1, \quad u > 0,\ v \geq 0, \quad 0 < r < 1,$$

with fixed foci ± 1. These may also be written in the form

$$|w + 1| - |w - 1| = 2r.$$

As $r \to 0^+$, the hyperbolic arcs converge to the boundary ray $[i0,\ i\infty]$.
As $r \to 1^-$, the hyperbolic arcs converge to the boundary ray $[1,\ \infty]$.

The images of the horizontal grid lines in the interior of the strip are the quarter-ellipses

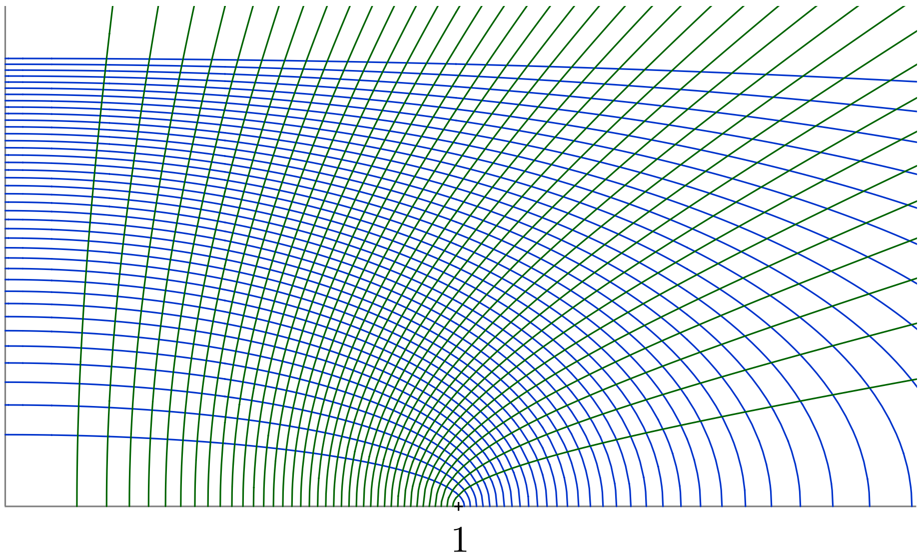
$$\frac{u^2}{r^2} + \frac{v^2}{r^2 - 1} = 1, \quad u,\ v \geq 0, \quad r > 1,$$

with fixed foci ± 1. These may also be written in the form

$$|w + 1| + |w - 1| = 2r.$$

As $r \to 1^+$, the elliptic arcs converge to the boundary segment $[0,\ 1]$.

Together the two families of arcs form a nonlinear orthogonal coordinate grid in the interior of the first quadrant of the w-plane.



Note that $\sin z$ is conformal at all points of the strip, except at the corner point $z = \frac{\pi}{2}$. Indeed, for every $z \neq \frac{\pi}{2}$ in the strip, $\sin'(z) = \cos z \neq 0$.