Smooth fit

Suppose we are given that \( f(x) = 2x \), for \( x \geq 1 \), and that \( f(x) = -x \), for \( x \leq -1 \).

Our goal is to join these two sides to create a function which is not just continuous but smooth. For this we need four conditions to be satisfied:

\[
\begin{align*}
f(1) &= 2, & f'(1) &= 2, \\
f(-1) &= 1, & f'(-1) &= -1.
\end{align*}
\]

Indeed, the values at the joints must agree and the slopes at the joints must agree.

What kind of function should \( f(x) \) be for \(-1 \leq x \leq 1\)?

Clearly \( f(x) = ax + b \) would not do the job. Let us try to patch the hole using a quadratic (if this does not work we could always try a higher-degree polynomial),

\[
f(x) = ax^2 + bx + c, \quad -1 \leq x \leq 1.
\]

The four conditions take the form

\[
\begin{align*}
f(1) &= a + b + c = 2, & f'(1) &= 2a + b = 2, \\
f(-1) &= a - b + c = 1, & f'(-1) &= -2a + b = -1.
\end{align*}
\]

These turn out to be easy to solve. Note that \( f'(1) + f'(-1) = 2b = 1 \), so \( b = 1/2 \).

Consequently, \( a = c = 3/4 \). So

\[
f(x) = \frac{3}{4} x^2 + \frac{1}{2} x + \frac{3}{4}, \quad -1 \leq x \leq 1.
\]

Here is how our solution looks like:

It is continuous and has no corners.