EXAM II ANSWERS

1. Consider the differential equation \( y''(x) = y(x) \).
   A) Find the characteristic roots.
      \( r^2 = 1, \quad r = \pm 1 \).
   B) Find a fundamental pair of solutions.
      \( y_1 = e^x, \quad y_2 = e^{-x} \).
   C) Describe all solutions with \( y(0) = 0 \).
      \[ y = ae^x + be^{-x}, \quad y(0) = a + b = 0. \] So \( y = (e^x - e^{-x}) \).
   D) Describe all solutions with \( y'(0) = 0 \).
      \[ y = ae^x + be^{-x}, \quad y'(0) = a - b = 0. \] So \( y = (e^x + e^{-x}) \).
   E) Find a fundamental pair of solutions \( y_1, y_2 \) such that \( y_1(0) = y_2'(0) = 1 \) and \( y_1'(0) = y_2(0) = 0 \).
      Since \( y_1'(0) = 0 \) we have \( y_1 = a(e^x + e^{-x}) \). Since \( y_1(0) = 1 \) we have \( a = 1/2 \).
      Since \( y_2(0) = 0 \) we have \( y_1 = b(e^x - e^{-x}) \). Since \( y_2'(0) = 1 \) we have \( b = 1/2 \).
      The functions \( y_1 = \cosh x \) and \( y_2 = \sinh x \) form a fundamental pair of solutions.

2. Let \( y''(x) + 4y'(x) + 4y(x) = 4x \).
   A) Solve the homogeneous equation.
      \( r^2 + 4r + 4 = (r + 2)^2 = 0, \quad r = -2, -2. \) So \( y = ae^{-2x} + bxe^{-2x} \).
   B) Find a particular solution.
      If \( y = Ax + B \) then \( A = 1 \) and \( B = -1. \) So \( y_p = x - 1 \).
   C) Solve the boundary value problem \( y(0) = 1, \quad y(1) = 0 \).
      \[ y = x - 1 + ae^{-2x} + bxe^{-2x} \] is the general solution. We have \( y(0) = -1 + a \) and \( y(1) = (a + b)e^{-2}. \) Hence \( a = 1, \quad b = -1 \) and \( y = (x - 1)(e^{-2x} - 1) \).

3. Let \( y''(x) + y(x) = 2 \cos x \).
   A) Find a solution by the method of undetermined coefficients.
      Observe that \( f(x) = 2 \cos x \) is a solution of the homogeneous equation. Look for \( y \) in the form \( y = x(A \cos x + B \sin x) \). Then \( A = 0, \quad B = 1 \) and \( y = x \sin x \).
   **Remark.** If \( y'' + y = 2 \cos x \) then \( y'''' + y'' = -2 \cos x \). Adding up we find that \( y'''' + 2y'' + y = 0 \). The characteristic equation \( r^4 + 2r^2 + 1 = (r^2 + 1)^2 = 0 \) has 2 double roots \( r = i, -i, \quad i, -i \). So \( y = A_1 \cos x + A_2 \cos x + B_1 \sin x + B_2 \sin x \).
      But the part \( A_1 \cos x + B_1 \sin x \) is the general solution of \( y'' + y = 0 \).
B) Find a solution using variation of parameters.

Solution 1. Standard variation. Seek $y$ in the form $y = A(x) \cos x + B(x) \sin x$, 
where $A' \cos x + B' \sin x = 0$. Then

$$\begin{cases} 
A' \cos x + B' \sin x = 0 \\
-A' \sin x + B' \cos x = 2 \cos x.
\end{cases}$$

Solve the system: $A' = -2 \sin x \cos x$, $B' = 2 \cos^2 x$. Integrate:

$$A = -\sin^2 x + c, \quad B = \int 2 \cos^2 x \, dx = \int (\cos(2x) + 1) \, dx = \frac{1}{2} \sin(2x) + x + c$$

Can take $A = -\sin^2 x$ and $B = \sin x \cos x + x$. Then $y = x \sin x$.

Solution 2. Seek $y$ in the form $y = A(x) \cos x + B(x) \sin x$. Then

$$y'' = A'' \cos x - A \cos x - 2A' \sin x + B'' \sin x - B \sin x + 2B' \cos x$$

So $(A'' - A + 2B') \cos x + (B'' - B - 2A' + B) \sin x = 2 \cos x$.

Or $(A'' + 2B') \cos x + (B'' - 2A') \sin x = 2 \cos x$.

Evidently, $A = 1$ and $B = x$ is a successful choice.

Solution 3. Reduction of order. Seek $y$ in the form $y = A(x) \cos x$. Then

$$y'' + y = A'' \cos x - 2A' \sin x. \quad \text{So} \quad A'' \cos x - 2A' \sin x = 2 \cos x.$$

Integrating factor: $\mu = \cos x$. We have $(A' \cos^2 x)' = 2 \cos^2 x$. So

$$A' \cos^2 x = \int 2 \cos^2 x \, dx = \int (\cos(2x) + 1) \, dx = \frac{1}{2} \sin(2x) + x + c$$

Can take $A' = \tan x + x \sec^2 x = (x \tan x)'$ and $A = x \tan x$. Then $y = x \sin x$.

Solution 4. Reduction of order. Seek $y$ in the form $y = A(x) \sin x$. Then

$$y'' + y = A'' \sin x + 2A' \cos x. \quad \text{So} \quad A'' \sin x + 2A' \cos x = 2 \cos x.$$ 

$A = x$ works.

4. Let $y'''(x) - 2y''(x) + y'(x) = 1 + 2e^x$.

A) Find a fundamental set of solutions of the homogeneous equation.

$$r''' - 2r'' + r = (r - 1)^2 r = 0.$$ 

So $r = 0, 1, 1$.

$y_1 = 1, \; y_2 = e^x, \; y_3 = xe^x$ form a fundamental set.

B) Check these solutions for linear independence.

$$\begin{vmatrix}
1 & e^x & xe^x \\
0 & e^x & e^x + xe^x \\
0 & e^x & 2e^x + xe^x
\end{vmatrix} = \begin{vmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 2
\end{vmatrix} = 1.$$

The Wronskian is nonzero, so functions are linearly independent.

C) Find the general solution.

The general solution is $y = y_p + c_0 + c_1 e^x + c_2 xe^x$. Seek a particular solution $y_p$
in the form $y_p = Ax + Bx^2 e^x$ and use superposition. Setting $y = Ax$ in

$$y'''(x) - 2y''(x) + y'(x) = 1$$
yields $A = 1$. Setting $y = Bx^2 e^x$ in

$$y'''(x) - 2y''(x) + y'(x) = 2e^x$$
yields $B = 1$. 