Conformality

Let $a, b \in \mathbb{C}$. Under what conditions is the mapping

$$f(z) = az + b\overline{z}$$

conformal? Clearly, $a \neq 0$ and $a + b \neq 0$ is necessary. Consider two straight segments

$$\gamma_1(t) = t, \quad 0 \leq t \leq 1,$$
$$\gamma_2(t) = tw, \quad 0 \leq t \leq 1,$$

forming an angle of $\arg w$ at the origin (w is a fixed nonzero complex number). Since

$$(f \circ \gamma_1)'(0) = (at + bt)' = a + b$$
$$(f \circ \gamma_2)'(0) = (awt + b\overline{w}t)' = aw + b\overline{w},$$

the images of γ_1 and γ_2 form an angle of $\arg(aw + b\overline{w}) - \arg(a + b)$ at the origin. The conformality condition

$$\arg(aw + b\overline{w}) - \arg(a + b) = \arg w$$

written as

$$\arg(a + b\overline{w}/w) = \arg(a + b)$$

thus forces $\arg(a + b\overline{w}/w)$ to be independent of w. But this is only possible if $b = 0$. The mapping $f(z) = az$, $a \neq 0$, is conformal and holomorphic.

Example. Consider the mapping

$$f(z) = z + 2\overline{z}$$

and let the curves γ_1 and γ_2 be as in the preceding example. Suppose that $w = i$.

What is the angle between γ_1 and γ_2 at the origin?

What is the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at the origin?

Is the angle preserved?