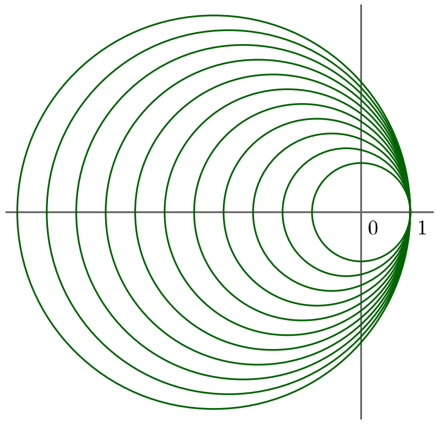
Geometric series

For $r > -1$, the nested open disks

$$D_r : \quad |z + r| < r + 1$$

are increasing with r and have a common boundary point of tangency $z = 1$.

As $r \to \infty$, these disks approach the half-plane $\Re z < 1$.



For z in D_r, the geometric power series

$$\sum_{n=0}^{\infty} \frac{1}{(1+r)^{n+1}} \, (z+r)^n = \frac{1}{1+r} \sum_{n=0}^{\infty} \left(\frac{z+r}{1+r} \right)^n$$

converges absolutely and locally uniformly to the same function $f(z) = \dfrac{1}{1-z}$,

which is holomorphic in $\mathbb{C} \setminus \{1\}$.

In particular, if $r = 0$, we have $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1.$