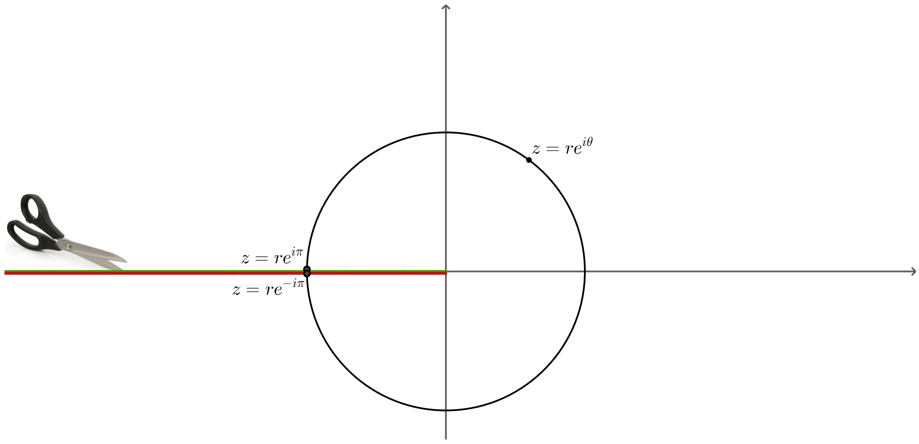
Complex Analysis

Grinshpan

Worksheet on $\arg z$, $\log z$, e^z, $\sqrt{z}$, z^2

The slit plane $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$ is an open connected subset of $\mathbb{C}$.
Consider the principal branch $\theta = \mathrm{Arg}\, z$ of $\arg z$ in $\mathbb{C}'$: $-\pi < \theta < \pi$.



As z starts on the lower edge of the cut and moves in a circular path around the origin to the upper edge of the cut, $\arg z$ changes continuously from $-\pi$ to π.

The branch of the logarithm $\ell(z) = \ln|z| + i\theta$ is holomorphic and univalent in $\mathbb{C}'$.
It maps $\mathbb{C}'$ onto the infinite horizontal strip $S: -\pi < \mathrm{Im}\, z < \pi$.
The function e^z restricted to S is the inverse of $\ell(z)$.

To check that $\ell(z)$ is holomorphic, use the Cauchy-Riemann equations in polar form:

$$ru_r = 1 = v_\theta \quad \text{and} \quad u_\theta = 0 = -rv_r.$$

To check that $\ell'(z) = 1/z$, use that $(\ln r)_x = x/r^2$ and $(\ln r)_y = y/r^2$:

$$\ell'(z) = u_x - iu_y = \frac{x}{r^2} - i\,\frac{y}{r^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z}.$$

$\ell(z)$ is conformal in $\mathbb{C}'$.

The square root branch $\sqrt{z} = \sqrt{|z|}\, e^{i\theta/2}$ is holomorphic and univalent in $\mathbb{C}'$.
It maps $\mathbb{C}'$ onto the right half-plane $H: \mathrm{Re}\, z > 0$.
The function z^2 restricted to H is the inverse of $\sqrt{z}$.

To check that $\sqrt{z}$ is holomorphic, note that $\sqrt{z} = e^{\frac{1}{2}\ell(z)}$, $z \in \mathbb{C}'$.
By the chain rule,

$$\left(\sqrt{z}\right)' = e^{\frac{1}{2}\ell(z)}\frac{1}{2z} = \frac{\sqrt{z}}{2z} = \frac{1}{2\sqrt{z}}.$$

$\sqrt{z}$ is conformal in $\mathbb{C}'$.

Can you suggest another branch of the square root in $\mathbb{C}'$? How many are possible?
How many branches of $\arg z$ are possible in $\mathbb{C}'$? Of $\log z$?