Complex Analysis

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Convergence: regular and uniform

The sequence $f_n(z)$ converges to $f(z)$ on a set $X \subseteq \mathbb{C}$ if, for each $z \in X$ and each $\varepsilon > 0$,

$$|f_n(z) - f(z)| < \varepsilon$$

for all but finitely many indices n. Equivalently, for each $z \in X$, $\lim\limits_{n\to\infty} |f_n(z) - f(z)| = 0$.

For different $z \in X$, $f_n(z)$ may approach $f(z)$ at different rate.

The sequence $f_n(z) = z^n$ converges to $f(z) = 0$ in the open unit disk $D: |z| < 1$.

Indeed, for each $z \in D$, $|z^n - 0| = |z|^n \to 0$, as $n \to \infty$.

The closer z is to the origin, the faster z^n tends to 0.

The closer z is to the edge of the disk, the slower z^n tends to 0.

The sequence $f_n(z)$ converges to $f(z)$ uniformly on a set $X \subseteq \mathbb{C}$ if, for each $\varepsilon > 0$,

$$|f_n(z) - f(z)| < \varepsilon$$

for all but finitely many indices n and all $z \in X$. Equivalently, $\lim\limits_{n\to\infty} \sup\limits_{z\in X} |f_n(z) - f(z)| = 0$.

The sequence $f_n(z) = z^n$ converges to $f(z) = 0$ uniformly on each $D_r: |z| < r$, $0 < r < 1$.

Indeed, given any $0 < r < 1$, we have $\sup\limits_{z \in D_r} |z^n - 0| = r^n \to 0$, as $n \to \infty$.

For each $z \in D_r$, z^n tends to 0 at least as fast as r^n.

The sequence $f_n(z) = z^n$ does not converge to $f(z) = 0$ uniformly on $D: |z| < 1$.

Reason: $\sup\limits_{z\in D} |z^n - 0| = 1 \not\to 0$, as $n \to \infty$.

For $z \in D$, z^n may tend to 0 arbitrarily slowly: given any $0 < \varepsilon < 1$, there does not exist an index n_0 such that $|z|^n < \varepsilon$ for all $n > n_0$ and all $z \in D$ at once.

The sequence $f_n(z) = z^n$ converges to $f(z) = 0$ locally uniformly in $D: |z| < 1$.

Each $z \in D$ has a neighborhood (for instance, any subdisk D_r with $|z| < r < 1$) such that z^n converges to 0 uniformly on that neighborhood.