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# SOBOLEV EXTENSION DOMAINS

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We shall discuss the following problem:

Given an open domain $\Omega \subset \mathbf{R}^d$, consider a Sobolev space $W_p^m(\Omega)$. What geometric conditions on $\Omega$ are necessary and sufficient for the possibility to extend every function $f \in W_p^m(\Omega)$ to a function $F \in W_p^m(\mathbf{R}^d)$?

One of the first deep results on this problem is due to Hassler Whitney, who has shown in 1934 that the equivalence of the geodesic metric on $\Omega$ to the usual Euclidean metric is sufficient to deal with the case $p = \infty$ for any $m \in \mathbf{N}$. Immediately after this result there were numerous attempts to prove that this geometric condition (called "quasiconvexity" by M. Gromov) is in fact necessary. This conjecture is often referred to as a "Whitney Conjecture".

It is easy to show that it is indeed necessary if $p = \infty$ and $m = 1$. However, the question was settled only in 1990s by the speaker, who gave a decisive answer: "It depends!"

More precisely, if $\Omega$ is a **planar finitely connected** domain, then for a fixed $m \in \mathbf{N}$ every function $f \in W_\infty^m(\Omega)$ can be extended to a function $F \in W_\infty^m(\mathbf{R}^2)$ if and only if the domain is quasiconvex. However, if one waives the topological condition of finite connectedness then the result is no longer true: for each $m \in \mathbf{N}$, $m > 1$, there exists a non-quasiconvex domain $\Omega_m \subset \mathbf{R}^2$ such that every function $f \in W_\infty^m(\Omega_m)$ is extendible to a function $F \in W_\infty^m(\mathbf{R}^2)$.

So in dimension 2 a topological restriction (finite connectedness) on the domain saves the Whitney Conjecture. However, in dimensions 3 and higher no topological restrictions would help: one can produce a counterexample for any given $m \in \mathbf{N}$, $m > 1$, where the domain is a topological ball, and even one can ensure that the boundary is smooth at all points except of one.

Recently, there was a substantial progress for Sobolev spaces $W_p^m(\Omega)$, $\Omega \subset \mathbf{R}^2$, $p > 2$. First, around 2005 Pavel Shvartsman proved (using a version of the initial Whitney construction), that if one replaces the geodesic metric by a special quasi-hyperbolic metric $\rho_p$ then a natural analog of quasiconvexity, called $p-$quasiconvexity, is sufficient for the extendability of functions from $W_p^m(\Omega)$, $\Omega \subset \mathbf{R}^d$, to the whole $\mathbf{R}^d$, with preservation of the class. In 2014 Shvartsman and the speaker were able to show that if $\Omega$ is a finitely connected planar domain, then $p-$quasiconvexity is also necessary for such an extendability.

The main tool in this work is an explicit construction of an almost fastest growing function in a simply connected planar domain, i.e., a function $f_{x,y} \in W_p^m(\Omega)$ which takes an almost largest possible value at $y \in \Omega$ provided that it is flat at $x \in \Omega$ and its $W_p^m$-norm is bounded by 1.

The main interest in all these considerations is in highly nontrivial geometric constructions which will be explained.