Inferences based on two samples

Example 9.1

\( m = 20 \) specimens \( \rightarrow \) sample average cold-rolled steel \( \bar{x} = 29.8 \) ksi

\( n = 25 \) specimens two-sided galvanized steel specimens \( \bar{y} = 34.7 \) ksi

\( N(\mu_1, \sigma_1^2) \quad N(\mu_2, \sigma_2^2) \)

\( \sigma_1^2 = 0.4 \quad \sigma_2^2 = 0.0 \)

\( H_0 : \mu_1 = \mu_2 \quad \Rightarrow \mu_1 - \mu_2 = 0 \)

\( H_a : \mu_1 \neq \mu_2 \quad \Rightarrow \mu_1 - \mu_2 \neq 0 \)

Hypothesis testing with significance level \( \alpha = 0.01 \)

\( \bar{x} - \bar{y} \)

Need to understand the behavior of
Theory: from a population

$X_1, \ldots, X_m$ – a random sample with mean $\mu_1$ and variance $\sigma_1^2$

$Y_1, \ldots, Y_n$ – a random sample from a population with mean $\mu_2$ and variance $\sigma_2^2$

The two samples are independent of each other.

Parameter of interest: $\mu_1 - \mu_2$

Estimator: $\bar{X} - \bar{Y}$

$E[\bar{X} - \bar{Y}] = E[\bar{X}] - E[\bar{Y}]$

$= \mu_1 - \mu_2$

$\text{Var}(\bar{X} - \bar{Y}) = 1^2 \text{Var}(\bar{X}) + (-1)^2 \text{Var}(\bar{Y})$

$= \frac{m}{m^2 \text{Var}(X_i)} + \frac{n}{n^2 \text{Var}(Y_j)}$

$= \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$

If $X_i \sim N(\mu_1, \sigma_1^2)$, $Y_j \sim N(\mu_2, \sigma_2^2)$

then $\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}})$
and

\[ Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1) \]

**Null hypothesis:** $H_0: \mu_1 - \mu_2 = \Delta_0$

**Test statistic value:**

\[ Z = \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \]

**Alternative hypotheses**

<table>
<thead>
<tr>
<th>Alternative hypotheses</th>
<th>Rejection region for level ( \alpha ) test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_a: \mu_1 - \mu_2 &gt; \Delta_0$</td>
<td>( z &gt; z_{\alpha} )</td>
</tr>
<tr>
<td>$H_a: \mu_1 - \mu_2 &lt; \Delta_0$</td>
<td>( z &lt; -z_{\alpha} )</td>
</tr>
<tr>
<td>$H_a: \mu_1 - \mu_2 \neq \Delta_0$</td>
<td>either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$</td>
</tr>
</tbody>
</table>

**Probability of type I error**

\[ P[ \text{type I error} ] = \alpha \]

$\Phi$  

$H_0$ is true but we reject $H_0$.
Type II error (when using a level $\alpha$ test)

$H_0: \mu_1 - \mu_2 = \Delta_0$
$H_a: \mu_1 - \mu_2 > \Delta_0$

$\beta(\Delta') = P[\text{not rejecting } H_0 \text{ when } \mu_1 - \mu_2 = \Delta']$

level
$\alpha$: we do not reject $H_0$

\[
\frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} < z_{\alpha}
\]

so $\beta(\Delta') = P[Z < z_{\alpha} \text{ when } \bar{X} - \bar{Y} \text{ has the distribution } N(\Delta', \sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}})]$

$= P[\bar{X} - \bar{Y} < \Delta_0 + \sigma_{\bar{X} - \bar{Y}} z_{\alpha} \text{ when } \mu_1 - \mu_2 = \Delta']$

\[
\frac{\bar{X} - \bar{Y} - \Delta'}{\sigma_{\bar{X} - \bar{Y}}}
\]

$\Delta_0 + \sigma_{\bar{X} - \bar{Y}} z_{\alpha} - \Delta' \overset{\text{N}(\Delta', \sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}})}{\sim} \bar{X} - \bar{Y}$

This means

$\frac{\Delta_0 + \sigma_{\bar{X} - \bar{Y}} z_{\alpha} - \Delta'}{\sigma_{\bar{X} - \bar{Y}}} \overset{\text{N}(0,1)}{\sim} \frac{\bar{X} - \bar{Y} - \Delta'}{\sigma_{\bar{X} - \bar{Y}}}$

$\beta(\Delta') = P[\frac{\bar{X} - \bar{Y} - \Delta'}{\sigma_{\bar{X} - \bar{Y}}} < \frac{\Delta_0 + \sigma_{\bar{X} - \bar{Y}} z_{\alpha} - \Delta'}{\sigma_{\bar{X} - \bar{Y}}}]$
Alternative Hypotheses

\[ \begin{align*}
    & \text{Ha: } \mu_1 - \mu_2 > \Delta_0 \quad \Phi \left( z_{\alpha} - \frac{\Delta' - \Delta_0}{\sigma} \right) \\
    & \text{Ha: } \mu_1 - \mu_2 < \Delta_0 \quad 1 - \Phi \left( -z_{\alpha} - \frac{\Delta' - \Delta_0}{\sigma} \right) \\
    & \text{Ha: } \mu_1 - \mu_2 \neq \Delta_0 \\
    & \quad \Phi \left( z_{\alpha/2} - \frac{\Delta' - \Delta_0}{\sigma} \right) \\
    & \quad \quad - \Phi \left( -z_{\alpha/2} - \frac{\Delta' - \Delta_0}{\sigma} \right) \\
\end{align*} \]

\[ \sigma = \sigma_{x-y} = \sqrt{\frac{\sigma_x^2}{m} + \frac{\sigma_y^2}{n}} \]
Example 9.3 (Example 9.1 continued)

Assume \( \mu_1 - \mu_2 = 5 \)

\[
\beta(5) = \Phi \left( \frac{2.58 - \frac{5-0}{1.34}}{1.34} \right) - \Phi \left( \frac{-2.58 - \frac{5-0}{1.34}}{1.34} \right)
\]

\[
= \Phi(-1.15) - \Phi(-6.31)
\]

\[
= 0.1251
\]

\[
P[ \text{not making a Type II error when } \mu_1 - \mu_2 = 5 ]
\]

\[
= 1 - \beta(5)
\]

\[
= 0.8749
\]

If the required probability is 0.9, a larger sample size should be used.
$P[\text{type I error}] = a \text{ specified } \alpha$

$P[\text{type II error}] = a \text{ specified } \beta$

when $\mu - \mu_0 = \Delta'$

If the alt. hypothesis is $H_a: \mu - \mu_0 > \Delta_0$

Equate: $\beta(\Delta') = \Phi \left( Z_\alpha - \frac{\Delta - \Delta_0}{\sigma} \right) = \beta$

then $Z_\alpha - \frac{\Delta - \Delta_0}{\sigma} = -Z_\beta$

so $Z_\alpha + Z_\beta = \frac{\Delta - \Delta_0}{\sigma}$

$\sigma^2 = \frac{(\Delta - \Delta_0)^2}{(Z_\alpha + Z_\beta)^2}$

Sample size condition found on page 366

If $n = m$, then

$$m = n = \frac{(\sigma_1^2 + \sigma_2^2)(Z_\alpha + Z_\beta)^2}{(\Delta - \Delta_0)^2}$$
Same if \( H_a: \mu_1 - \mu_2 < \Delta_0 \)

If \( H_a: \mu_1 - \mu_2 \neq \Delta_0 \), replace \( \Delta \) by \( \frac{\Delta}{2} \).
Example

78 observations on strength of \( \frac{3}{16} \)-in. diameter bolt

\[
\text{sample mean} = 4.250 \\
\text{Sample Std} = 1.30
\]

88 observations on strength of \( \frac{1}{2} \)-in. diameter bolt

\[
\text{sample mean} = 7.140 \\
\text{Sample Std} = 1.680
\]

A 95% CI for \( \mu_1 - \mu_2 \) is

\[
\bar{x}_1 - \bar{x}_2 \pm z_{0.025} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}
\]

\[
4.25 - 7.14 \pm 1.96 \sqrt{\frac{1.30^2}{78} + \frac{1.68^2}{88}}
\]

\[
= (-3.34, -2.04)
\]

Interpretation: With 95% confidence, \( \mu_2 \) exceeds \( \mu_1 \) by between \( -3.34 \) and \( -2.04 \) kip.

ave. strength of \( \frac{3}{16} \)-in. bolt \qquad ave. strength of \( \frac{1}{2} \)-in. bolt \,
\qquad \frac{3}{16} \text{-in. bolt} \quad \qquad 3.34 \text{ kip.}
Section 9.2

The two-sample t-test and CI.

Setup: Both populations are normal
i.e. we have two random samples
\[ X_1, \ldots, X_m \sim i.i.d. N(\mu_1, \sigma_1^2) \]
\[ Y_1, \ldots, Y_n \sim i.i.d. N(\mu_2, \sigma_2^2) \]

From Ch 7,
\[ \frac{X - \mu_1}{S_1 \sqrt{m}} \]
has the t-distribution with \( m-1 \) df.
\[ \frac{Y - \mu_2}{S_2 \sqrt{n}} \]
has the t-distribution with \( n-1 \) df. \( \sigma_1, \sigma_2 \) are unknown.

\( S_1, S_2 \) are sample stds, NOT \( \sigma_1, \sigma_2 \).

Thm. The random variable
\[ T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}} \]
has approximately a t-distribution with \( \nu \) df, estimated from the data by
\[ \nu = \frac{(S_1^2/m + S_2^2/n)^2}{\frac{(S_1^2/m)^2}{m-1} + \frac{S_2^2/n^2}{n-1}} \]
where
\[ se_1 = \frac{S_1}{\sqrt{m}} \quad se_2 = \frac{S_2}{\sqrt{n}} \]
(Round \( \nu \) down to the nearest integer.)

This theorem allows us to derive CI for \( \mu_1 - \mu_2 \), and also a test statistic for \( \mu_1 - \mu_2 \) in hypothesis testing.

CI: The two sample \( t \) confidence interval for \( \mu_1 - \mu_2 \) with confidence level \( 100(1-\alpha)\% \) is

\[
\bar{x} - \bar{y} \pm t_{\alpha/2, \nu} \sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}
\]

where \( \nu \) is given by \( \star \).

Q: What about one-sided CI?

Hyp.: The two-sample \( t \) test for testing testing

\[ H_0 : \mu_1 - \mu_2 = \Delta_0 \] is as follows:

<table>
<thead>
<tr>
<th>Alt. Hypothesis</th>
<th>rejection region for approx. level ( \alpha ) test</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_a : \mu_1 - \mu_2 &gt; \Delta_0 )</td>
<td>( t &gt; t_{\alpha, \nu} )</td>
</tr>
<tr>
<td>( H_a : \mu_1 - \mu_2 &lt; \Delta_0 )</td>
<td>( t \leq -t_{\alpha, \nu} )</td>
</tr>
<tr>
<td>( H_a : \mu_1 - \mu_2 \neq \Delta_0 )</td>
<td>( t &gt; t_{\alpha/2, \nu} ) or ( t \leq -t_{\alpha/2, \nu} )</td>
</tr>
</tbody>
</table>

Test statistic value

\[
t = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}
\]
Example 9.6  Air permeability of woven textile fabrics.

<table>
<thead>
<tr>
<th>Fabric type</th>
<th>Sample size</th>
<th>Sample mean</th>
<th>Sample std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cotton</td>
<td>10</td>
<td>51.71</td>
<td>0.79</td>
</tr>
<tr>
<td>Triacetate</td>
<td>10</td>
<td>136.14</td>
<td>3.59</td>
</tr>
</tbody>
</table>

Want 95% CI for $\mu_1 - \mu_2$. Sample sizes too small for CLT → use $t$ CI.

$$df = \frac{\left(\frac{0.6241}{10} + \frac{12.8881}{10}\right)^2}{\left(\frac{0.6241/10}{9}\right) + \left(\frac{12.8881/10}{9}\right)} = 9.87$$

Use $v = 9$, $t_{0.025, 9} = 2.262$ (Appendix table A.5)

Resulting CI is

$$51.71 - 136.14 \pm (2.262) \sqrt{\frac{0.6241}{10} + \frac{12.8881}{10}}$$

$$= (-87.06, -81.80).$$
(Section 9.3)

Paired data

\[ X_1 \ldots X_n \} \text{ dependent} \]

\[ Y_1 \ldots Y_n \]

e.g.

6 locations

river in South India

collect zinc concentration in bottom water \((x)\)

zinc concentration in surface water \((y)\)

<table>
<thead>
<tr>
<th>Location</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>0.430</td>
<td>0.266</td>
<td>0.567</td>
<td>0.531</td>
<td>0.707</td>
<td>0.716</td>
</tr>
<tr>
<td>(y)</td>
<td>0.415</td>
<td>0.238</td>
<td>0.390</td>
<td>0.410</td>
<td>0.605</td>
<td>0.609</td>
</tr>
<tr>
<td>(d)</td>
<td>0.015</td>
<td>0.028</td>
<td>0.174</td>
<td>0.121</td>
<td>0.102</td>
<td>0.107</td>
</tr>
</tbody>
</table>
Assume $X_i \sim N(\mu_i, \sigma_i^2)$, $Y_i \sim N(\mu_2, \sigma_2^2)$

$E(X_i) = \mu$, $E(Y_i) = \mu_2$

$D_i = X_i - Y_i$

$D_i \sim N(\mu_0, \sigma_0^2)$

$\mu_1 - \mu_2$

**KEY POINT:** $X_i, Y_i$ are not independent.

Hypothesis testing on $\mu_1 - \mu_2$ cannot be based on the two-sample $t$-test.

Instead, apply a one-sample $t$-test to $D_i$'s.

Aside:

$\text{var}(X-Y)$ is no longer $\text{var}(X) + \text{var}(Y)$

$$\text{var}(X-Y) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n} + \frac{\text{Cov}(XY)}{n}$$

But

$$\frac{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2}{n}$$
The paired t-test

Null hypothesis: \( H_0 : \mu_D = \Delta_0 \) \( (D = X - Y) \)

Test statistic: \( t = \frac{\overline{d} - \Delta_0}{S_D / \sqrt{n}} \)

- \( \overline{d} \) = sample mean of \( d_i \)'s
- \( S_D \) = sample std. of \( d_i \)'s

Alternative hypothesis

- \( H_a : \mu_D > \Delta_0 \) \( t > t_{\alpha, n-1} \)
- \( H_a : \mu_D < \Delta_0 \) \( t \leq -t_{\alpha, n-1} \)
- \( H_a : \mu_D \neq \Delta_0 \) \( t \geq t_{\alpha/2, n-1} \) or \( t \leq -t_{\alpha/2, n-1} \)
Example

<table>
<thead>
<tr>
<th>Subject</th>
<th>Before</th>
<th>After</th>
<th>Difference</th>
<th>$d$</th>
<th>$S_d$</th>
<th>$8d$</th>
<th>$H_0: \mu_d = 0$, $H_1: \mu_d \neq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>6.75</td>
<td>8.234</td>
<td>6.75</td>
<td>$t = \frac{\bar{d} - \Delta_0}{S_d/\sqrt{n}} = \frac{6.75}{8.234/\sqrt{16}} = 3.28$</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
<td>8</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td>$\approx 3.3$</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>+area $= 0.002$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>19</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td>$p\text{-value} \approx 2 \cdot (0.002) = 0.004$.</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>19</td>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td>$\leq 0.004$.</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>19</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td>$\geq 0.004$.</td>
</tr>
</tbody>
</table>

**Interpretation:**

$H_0$ can be rejected at any significance level $\geq 0.004$. 
The paired t CI for \( \mu_d \) is

\[
\bar{d} \pm t_{\alpha/2, n-1} \cdot S_d / \sqrt{n}
\]

Difference between 2 population proportions

\begin{align*}
\% \text{age good men} &= p_1 \\
\% \text{age good women} &= p_2
\end{align*}

\( X_1, \ldots, X_m \sim \text{iid Bernoulli} (p_1) \)
\( Y_1, \ldots, Y_n \sim \text{iid Bernoulli} (p_2) \)

Estimator for \( p_1 - p_2 \) : \( \hat{p}_1 - \hat{p}_2 \), \( \frac{\text{EC}}{\text{VC}} \) \( \frac{3}{3} \)
**Fact:** \[ E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2 \]

\[ V(\hat{p}_1 - \hat{p}_2) = \frac{p_1 \hat{p}_1}{n} + \frac{p_2 \hat{p}_2}{n} \]

where \( \hat{p}_i = l - p_i \).

**Proof:**

\[ E[x_i] = np_1 \]
\[ E[y_i] = np_2 \]

\[ E\left[ \frac{\hat{X}}{n} - \frac{\hat{Y}}{n} \right] \xrightarrow{\text{weak}} \frac{1}{m} E[X_i] - \frac{1}{n} E[Y_i] \]

\[ \frac{1}{m} = \hat{p}_1 - \hat{p}_2 = \frac{1}{m} (np_1) - \frac{1}{n} (np_2) \]

\[ \Rightarrow \hat{p}_1 - \hat{p}_2 = p_1 - p_2 \]

\[ V\left[ \frac{\hat{X}}{n} - \frac{\hat{Y}}{n} \right] = \left( \frac{1}{m} \right)^2 V[x_i] + \left( \frac{1}{n} \right)^2 V[y_i] \]

\[ = \frac{m p_1 \hat{p}_1}{m^2} + \frac{np_2 \hat{p}_2}{n^2} = l - p_2 \]

\[ = \frac{p_1 \hat{p}_1}{m} + \frac{p_2 \hat{p}_2}{n} \]

If \( m, n \) large, CLT \( \Rightarrow \)

\[ \hat{p}_1 - \hat{p}_2 \text{ is approximately } N(p_1 - p_2, \sqrt{\frac{p_1 \hat{p}_1}{m} + \frac{p_2 \hat{p}_2}{n}}) \]
A large sample test procedure.

Assume $H_0 : p_1 - p_2 = 0$

If $H_0$ is true,

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m} + \frac{1}{n}\right)}}$$

where $p_1 = p_2 = p$ when $H_0$ is true.

 Approximately $N(0,1)$

$m, n$ large

don't know $p$

cannot use $Z$ as the test statistics,

\[ \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{m} + \frac{1}{n}\right)}} \]

$w_1 + w_2 = 1$

approximately $N(0,1)$ when $m, n$ large

$\hat{p}_1, \hat{p}_2$ are the sample proportions.
H0: \( p_1 - p_2 \geq 0 \)  
H1: \( p_1 - p_2 > 0 \)  
H2: \( p_1 - p_2 < 0 \)  
H3: \( p_1 - p_2 \neq 0 \)  

\[ z \geq \frac{z}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \]  
\[ \text{or} \quad z \leq -\frac{z}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \]

\( p \)-value similar...
Example 9.11

<table>
<thead>
<tr>
<th>Plead</th>
<th>guilty</th>
<th>not Guilty</th>
</tr>
</thead>
<tbody>
<tr>
<td># judged guilty</td>
<td>m=191</td>
<td>n=64</td>
</tr>
<tr>
<td>number sentenced to prison</td>
<td>x=101</td>
<td>y=56</td>
</tr>
<tr>
<td>Sample proportion</td>
<td>( \hat{p}_1 = 0.529 )</td>
<td>( \hat{p}_2 = 0.875 )</td>
</tr>
</tbody>
</table>

\( H_0 : \hat{p}_1 - \hat{p}_2 = 0 \), \( H_a : \hat{p}_1 - \hat{p}_2 \neq 0 \)

\[ \hat{p} = \frac{x+y}{m+n} = \frac{101+56}{191+64} = 0.616 \]

\[ z = \frac{0.529 - 0.875}{\sqrt{(0.616)(0.384)(\frac{1}{191} + \frac{1}{64})}} = -4.94 \]

Rejection region for \( \alpha = 0.01 \):

\[ z_{\text{crit}} = 2.58 \quad \text{or} \quad z \leq -2.58 \]

Since \(-4.94 \leq -2.58\), we reject \( H_0 \).

Plead guilty + Judged guilty

Plead NOT guilty + Judged guilty

```
I am "sinful"
Jail
```

```
I am "innocent"
Jail
```

\[ 0.14 \]

\[ 0.5 \]
H0: \( \mu_1 - \mu_2 = 0 \) 

H1: \( \mu_1 - \mu_2 > 0 \) 

\[
\Phi \left( \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}} \right)
\]

H1: \( \mu_1 - \mu_2 < 0 \) 

\[
1 - \Phi \left( \frac{-\hat{p}_1 + \hat{p}_2}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}} \right)
\]

H1: \( \mu_1 - \mu_2 \neq 0 \) 

\[
\Phi \left( \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}} \right) - \Phi \left( \frac{-\hat{p}_1 + \hat{p}_2}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}} \right)
\]

\[\bar{p} = \frac{mp_1 + np_2}{m+n}\]

\[\bar{q} = \frac{mq_1 + nq_2}{m+n} \quad (= 1-\bar{p})\]

In above 

\[\sigma = \sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}\]

Sample size determination:

For specified \( p_1, p_2 \) with \( \mu_1 - \mu_2 = \delta \)

\( \alpha, \beta \),

H1: \( \mu_1 - \mu_2 > 0 \) 

condition on \( m, n \):

\[
\frac{1}{\delta} \left[ \frac{\alpha}{\sqrt{\bar{p} \bar{q} \left( \frac{1}{m} + \frac{1}{n} \right)}} - (\hat{p}_1 - \hat{p}_2) \right] = -z_\beta
\]
If \( \min_n = \circ \),

\[
\begin{align*}
\text{If } H_a: & \ p_1 - p_2 < 0 \\
\text{If } H_a: & \ p_1 - p_2 \neq 0, \text{ replace } d \text{ by } \frac{d}{2} \text{ in } (\circ) \text{.}
\end{align*}
\]

Read Example 9.12.
Large Sample

CI for $p_1 - p_2$?

$\hat{P}_1 - \hat{P}_2 \sim N(p_1 - p_2, \sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}})$

approximate

100(1-α)% CI for $p_1 - p_2$:

\[
\frac{\hat{P}_1 - \hat{P}_2 - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}} \sim N(0, 1)
\]

\[
\frac{\hat{P}_1 - \hat{P}_2 - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}} \sim N(0, 1)
\]

area = 1-α

~ $Z_{0.42}$

~ $Z_{0.42}$
\[ P \left[ -\frac{z_{1-\alpha}}{2} \leq \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}} \leq \frac{z_{1-\alpha}}{2} \right] \approx 1 - \alpha \]

\[ P \left[ \hat{p}_1 - \hat{p}_2 \leq p_1 - p_2 \leq \hat{p}_1 - \hat{p}_2 + \frac{z_{1-\alpha}}{2} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}} \right] \approx 1 - \alpha \]

CI:

\[ p_1 - p_2 \pm \frac{z_{1-\alpha}}{2} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}} \]