Cutting Corners on the Sphere

Thomas P.-Y. Yu

Abstract. We show that the α-corner cutting scheme, when adapted to the n-sphere based on geodesics, produces curves with critical Hölder regularity the same as that of the corner cutting scheme in the regular Euclidean setting. In other words, the 'curviness' of the sphere does not deteriorate (or improve) the smoothness of the corner cutting scheme at all. On the one hand, this result proves a special case of the so-called smoothness equivalence conjectures pertaining to the manifold-valued data refinement schemes in [11]; on the other hand, our result contrasts in an unexpected way with seemingly similar results by Noakes [7, 9].

Dedication: To the memory of my mother (1935-2004), who got me addicted to corners and spheres.

§1. Introduction

This work is inspired by the recent article of Ur Rahnam *et al* [11] on multiscale representation for time series or spatial arrays that take values on a manifold, as well as by the analysis of manifold subdivision schemes due to Wallner and Dyn [14]. In [11], a novel wavelet-like representation is proposed for data of the form

$$f : [0,1]^s \to \mathcal{M}, \tag{1}$$

where $\mathcal{M}$ can be any symmetric Riemannian manifold such as the sphere S^{n-1}, the Grassman manifold $G(n,k)$, and the rotation group $SO(n)$. The starting point of their method is the well-known connection of biorthogonal wavelets with subdivision schemes used in a prediction and correction manner [2, 3]. Underlying their approach is a nonlinear refinement subdivision scheme which operates as follows: Given $u_{j,k} \in \mathcal{M}$

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that is in an appropriate sense the averages of a reasonably smooth signal $f : \mathbb{R} \to \mathcal{M}$ over the scale j dyadic cells

$$I_{j,k} := [2^{-j}k, 2^{-j}(k+1)],$$

one can use them to predict the averages of f over the finer dyadic cells $I_{j+1,k}$ based on somehow "fitting a polynomial" to the data $u_{j,k-L}, \ldots, u_{j,k}, \ldots, u_{j,k+L}$, and then using $\widetilde{u}_{j+1,2k}$ and $\widetilde{u}_{j+1,2k+1}$, the "averages of the polynomial" at $I_{j+1,2k}$ and $I_{j+1,2k+1}$, as the estimands of $u_{j+1,2k}$ and $u_{j+1,2k+1}$, respectively. This "procedure" gives a nonlinear subdivision scheme on the one hand, and allows for a kind of wavelet decomposition based on the "differences" between $u_{j+1,k}$ and $\widetilde{u}_{j+1,k}$ on the other hand. Of course, on a nonlinear manifold the above phrases in double quotes make no immediate mathematical sense; the suggestions in [11] are to use the logarithmic map to project all the points

$$u_{j,k-L}, \ldots, u_{j,k}, \ldots, u_{j,k+L}, u_{j+1,2k}, u_{j+1,2k+1} \in \mathcal{M}$$

onto the tangent plane $T_{u_{j,k}}\mathcal{M}$, and then the corresponding operations on the tangent plane – a *linear* space – become well-defined. The exponential map is used to send the predicted points on the tangent plane $T_{u_{j,k}}\mathcal{M}$ back to the manifold. As such, the "wavelet coefficients" of f take on values on the tangent planes of $\mathcal{M}$; whereas the "scaling coefficients" of f take on values on the manifold itself. See [11] for more details. Related constructions of nonlinear pyramid transforms, motivated by different practical reasons, are proposed in [4, 10].

Since an aforementioned wavelet transform for manifold valued data is structurally the same as that of an average-interpolating wavelet transform [3] for standard real valued data, the following natural theoretical questions arise:

I. Does the 'curviness' of $\mathcal{M}$ affect the decay properties of wavelet coefficients when applied to piecewise smooth signals $f : \mathbb{R} \to \mathcal{M}$?

II. Does the 'curviness' of $\mathcal{M}$ affect the smoothness of the underlying subdivision scheme?

These questions are addressed to some extent in [11]. In particular, it is claimed in [11, Section 3.4.3] that the answer to question I is affirmative, provided that $\mathcal{M}$ is a smooth enough manifold; and they conjecture that question II has an equally affirmative answer. On the other hand, it is shown in [14] that, for certain kinds of manifold subdivision schemes (not based on the "Log-Exp linearization" in [11] but still constructed from an underlying linear subdivision scheme), if the underlying linear subdivision scheme is C^1, and also that the nonlinear scheme and the linear scheme satisfy a certain proximity condition, then the nonlinear scheme is C^1 as well.

Modulo the difference in the linearization methods, the results of Wallner and Dyn answer Question II affirmatively – but in a conservative way. Indeed, it is conceivable that the 'curviness' of a manifold may 'take away' at least a bit of the smoothness (if smoothness is measured using, say, fractional Hölder exponent) from that of the underlying linear subdivision scheme; yet numerical computations suggest that this does not happen [11, 17]. See Section also.

In this paper, we pursue a precise case study of the second question and show that in the case of $\mathcal{M} = S^{n-1}$ and the underlying linear subdivision scheme being the so-called α-corner cutting scheme, the nonlinear subdivision scheme has exactly the same critical Hölder regularity as that of the linear scheme.

The main result in this note is of the same flavor as those in [16, 15], namely, a nonlinear subdivision scheme shares the same critical Hölder regularity as that of an approximating linear scheme; the related article [18] shows that one should not take such smoothness equivalence results for granted: there are (non-pathological) nonlinear subdivision schemes that – *unlike any linear scheme* – produce curves with smoothness exponents that are heavily dependent on the initial data.

§2. Cutting Corners on the Sphere

Set $\alpha \in (0, 1/2)$. The α-corner cutting scheme on a Riemannian manifold $\mathcal{M}$ is defined by

$$u_{j+1,2k-1} = \text{g-av}_\alpha(u_{j,k-1}, u_{j,k}), \quad u_{j+1,2k} = \text{g-av}_{1-\alpha}(u_{j,k-1}, u_{j,k}). \tag{2}$$

Here $\text{g-av}_\alpha(u, v) := g(\alpha)$, where $g : [0, 1] \to \mathcal{M}$ is the unique geodesic that connects u and v when u and v are close enough, is the geodesic averaging operator introduced in [14, 8].

In the case when $\mathcal{M}$ is a Riemannian manifold, the above scheme coincides with the manifold-refinement scheme in [11] based on the "Log-Exp linearization".

We work with $\mathcal{M} = S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ viewed as a regular hypersurface in $\mathbb{R}^n$. In order for all the geodesic averages in (2) to be well-defined on S^{n-1}, it is necessary and sufficient that the initial data $u_{0,k}$ is such that for no $k \in \mathbb{Z}$, $u_{0,k-1}$ and $u_{0,k}$ form an antipodal pair. Then, by (8)-(9) below, the same holds of u_j for any $j > 0$. We write T or T_α to denote the α-corner cutting scheme on S^{n-1}.

We denote by S_α or S the corresponding linear α-corner-cutting scheme operating on $\mathbb{R}^n$, which is simply (2) with the geodesic averages $\text{g-av}_\alpha(u, v)$ replaced by the standard convex combination $(1-\alpha)\,u + \alpha\, v$. Recall that $S_{1/4}$ is the well-known Chaikin's corner cutting algorithm, and was first shown by Riesenfeld to generate quadratic B-splines.

More formally, we think of S as a linear operator on $[\ell(\mathbb{Z})]^n := \{v \mid v : \mathbb{Z} \to \mathbb{R}^n\}$ and T as a nonlinear operator on

$$\{v : \mathbb{Z} \to S^{n-1} \mid v_k \neq -v_{k+1},\ \forall\, k\} \subset [\ell(\mathbb{Z})]^n.$$

So (2) may be expressed as $Tu_j = u_{j+1}$, $j = 0, 1, \ldots$, and similarly for S. (If given a finite initial control polygon $u_{0,k}$, $k = 0, \ldots, N$, we can simply extend it to $(u_{0,k})_{k\in\mathbb{Z}}$ by letting $u_{0,k} = u_{0,1}$, for $k < 0$, and $u_{0,k} = u_{0,N}$, for $k > N$.)

We call a linear or nonlinear subdivision operator R convergent if for any bounded initial data u_0 there exists a continuous $f_{u_0} : \mathbb{R} \to \mathbb{R}^n$ such that

$$\lim_{j\to\infty} \max_{k\in\mathbb{Z}} \left\| f(2^{-j}k) - (R^j u_0)_k \right\| = 0.$$

In this case, we also define the critical Hölder regularity of R by

$$s_\infty(R) := \inf_{u_0} \sup \left\{ \alpha : f_{u_0} \in \operatorname{Lip} \alpha \right\}.$$

(There is no difference in saying $f : \mathbb{R} \to \mathbb{R}^n$ or $f : \mathbb{R} \to S^{n-1}$ when discussing the smoothness of f, as S^{n-1} is regularly embedded in $\mathbb{R}^n$.)

Theorem 1. *For $\alpha \in (0, 1/2)$,*

$$s_\infty(T_\alpha) = s_\infty(S_\alpha) = \begin{cases} -\log_2(\max(2\alpha, 1-2\alpha)) < 1, & \text{if } \alpha \neq 1/4 \\ 2, & \text{if } \alpha = 1/4 \end{cases}. \quad (3)$$

Proof:
0° It is a standard calculation of subdivision theory to determine $s_\infty(S_\alpha)$. We use specifically the results from the paper [12]. The symbol the subdivision mask of S is

$$a(z) = \alpha + (1-\alpha)z + (1-\alpha)z^2 + \alpha z^3 = (1+z)(\alpha z^2 + (1-2\alpha)z + \alpha).$$

So there exists subdivision operator $S^{[1]}$ which satisfies $S^{[1]} \circ \Delta = \Delta \circ S$ with

$$\|S^{[1]}\|_{\ell^\infty} = \rho(S^{[1]}) = \max(\alpha + \alpha, 1 - 2\alpha) =: \gamma\,;$$

and

$$\|\Delta\, S^j\, \delta\|_{\ell^\infty} \asymp \gamma^j. \quad (4)$$

(Here Δ stands for the forward differencing operator.) This already implies $s_\infty(S) \geq -\log\gamma$. The fact that $\gamma < 1/2$ if $\alpha \neq 1/4$ and that the mask satisfies a stability condition (see below) implies that $s_\infty(S_\alpha) = -\log_2\gamma$. When $\alpha = 1/4$, $a(z) = (1+z)^3/4$ which is, of course, the mask of the C^1 quadratic B-Spline, so $s_\infty(S_{1/4}) = 2$. (Without the spline connection, one also sees from the factorization that

$$\|\Delta^r S^j_{1/4}\, \delta\|_{\ell^\infty} \asymp (1/4)^j \text{ for } r = 2 \text{ or } 3.$$

Together with the stability condition below, we can conclude again that $s_\infty(S_{1/4}) = 2$.)

The refinable function φ (i.e. the subdivision limit of the Kronecker delta sequence δ) associated with this subdivision scheme is supported at $[-1, 2]$. One stability condition we can use is that of [12, Section 6], which requires that for some $x \in \mathbb{R}$,

$$\sum_n \varphi(n+x)\, e^{in\omega} \neq 0, \quad \forall\, \omega.$$

Using eigen-analysis, one easily calculates that $\varphi(1/2) = 1-\alpha$, $\varphi(-1/2) = \varphi(3/2) = \alpha/2$, so $\sum_n \varphi(n+1/2)\, e^{in\omega} = (1-\alpha) + \alpha\cos(\omega) = 1 - \alpha(1 - \cos(\omega)) > 0$, $\forall\, \alpha \in (0, 1/2)$.

We divide the rest of the proof into four parts.

1° For $u, v \in S^{n-1}$, $u \neq -v$, we have

$$\text{g-av}_\beta(u, v) = \omega(\theta, 1-\beta)\, u + \omega(\theta, \beta)\, v, \tag{5}$$

where $\theta = \angle(u, v) = \cos^{-1}\langle u, v\rangle \in [0, \pi)$ and

$$\omega(\theta, \beta) := \begin{cases} \dfrac{\sin(\theta\beta)}{\sin\theta}, & \text{if } \theta \in (0, \pi) \\ \beta, & \text{if } \theta = 0 \end{cases}. \tag{6}$$

Define $\theta_{j,k} := \angle(u_{j,k}, u_{j,k+1})$. Then (2) becomes

$$\begin{aligned} u_{j+1,2k-1} &= \omega(\theta_{j,k-1}, 1-\alpha)\, u_{j,k-1} + \omega(\theta_{j,k-1}, \alpha)\, u_{j,k}, \\ u_{j+1,2k} &= \omega(\theta_{j,k-1}, \alpha)\, u_{j,k-1} + \omega(\theta_{j,k-1}, 1-\alpha)\, u_{j,k}. \end{aligned} \tag{7}$$

2° Denote by $\widetilde{d}(u, v)$ the geodesic distance between u and v; since we are on the sphere, $\widetilde{d}(u, v) = \angle(u, v)$. Observe that

$$\theta_{j+1,2k-1} = (1-2\alpha)\, \theta_{j,k-1}, \tag{8}$$

$$\begin{aligned} \theta_{j+1,2k} &= \widetilde{d}(u_{j+1,2k}, u_{j+1,2k+1}) \\ &\leq \widetilde{d}(u_{j+1,2k}, u_{j,k}) + \widetilde{d}(u_{j,k}, u_{j+1,2k+1}) = \alpha\,(\theta_{j,k-1} + \theta_{j,k}); \end{aligned} \tag{9}$$

see also Figure 1. So

$$\theta_{j,k} = O(\gamma^j), \quad \text{where} \quad \gamma = \max(2\alpha, 1-2\alpha). \tag{10}$$

Since

$$\angle(u, v) = \widetilde{d}(u, v) \geq \|u - v\|, \tag{11}$$

(10) gives

$$\max_k \|u_{j,k} - u_{j,k-1}\| = O(\gamma^j). \tag{12}$$

Note that $\gamma < 1/2$ unless $\alpha = 1/4$.
3° We use the perturbation argument in [1]. By [1, Theorem 3.3], if there exists $C > 0$ and $\nu > 0$ such that

$$\|(T-S)\, T^j u\|_{\ell^\infty} \leq C\, ||u||_{\ell^\infty}\, 2^{-j\nu}, \tag{13}$$

then T is convergent and

$$s_\infty(T) \geq \min(\nu, s_\infty(S)).$$

To bound $\|(T-S)u_j\|_{\ell^\infty}$, notice that by (7) and that $\|u_{j,k}\| = 1$, we obtain

$$\begin{aligned} &\left\|(Tu_j)_{2k} - (Su_j)_{2k}\right\| \\ &= \left\|[\omega(\theta_{j,k-1},\alpha) - \alpha]\, u_{j,k-1} + [\omega(\theta_{j,k-1}, 1-\alpha) - (1-\alpha)]\, u_{j,k}\right\| \\ &\leq \ |\, \omega(\theta_{j,k-1},\alpha) - \alpha\,| + |\, \omega(\theta_{j,k-1}, 1-\alpha) - (1-\alpha)\,|\ ; \end{aligned} \tag{14}$$

and the same bound holds for $\|(Tu_j)_{2k-1} - (Su_j)_{2k-1}\|$.
For small $\theta > 0$,

$$\omega(\theta,\beta) - \beta = \frac{\sin(\theta\beta)}{\sin\theta} - \beta = O(\theta^2). \tag{15}$$

Combining (15), (14) and (10), we have

$$\|(T-S)\, T^j u\|_{\ell^\infty} = O(\gamma^{2j}). \tag{16}$$

So by the perturbation theorem, we conclude that

$$s_\infty(T_\alpha) \geq \min\left(-2\log_2(\gamma), s_\infty(S_\alpha)\right) = \begin{cases} -\log_2\gamma, & \text{if } \alpha \neq 1/4 \\ 2, & \text{if } \alpha = 1/4 \end{cases},$$

which is the right-hand side of (3).
4° It remains to show $s_\infty(T_\alpha) \leq s_\infty(S_\alpha)$. Simply consider initial data points on the sphere $(u_{0,k})_{k\in\mathbb{Z}}$ that lie on the same great circle, for example

$$u_{0,k} = [\cos(\theta_{0,k}), \sin(\theta_{0,k}), 0, \ldots, 0]^T,$$

where $\theta_{0,k}$ is monotone with $0 \leq \theta_{0,k+1} - \theta_{0,k} < \pi$ for all k, then the limit curve produced by T_α is of the form

$$f_{u_0}(t) = [\cos(\widetilde{\theta}(t)), \sin(\widetilde{\theta}(t)), 0, \ldots, 0]^T,$$

where $\widetilde{\theta}(t)$ is the limit function obtained from applying S_α to the initial data $(\theta_{0,k})_{k\in\mathbb{Z}}$. Choose $\theta_{0,k}$ so that the critical Hölder regularity of $\widetilde{\theta}(t)$

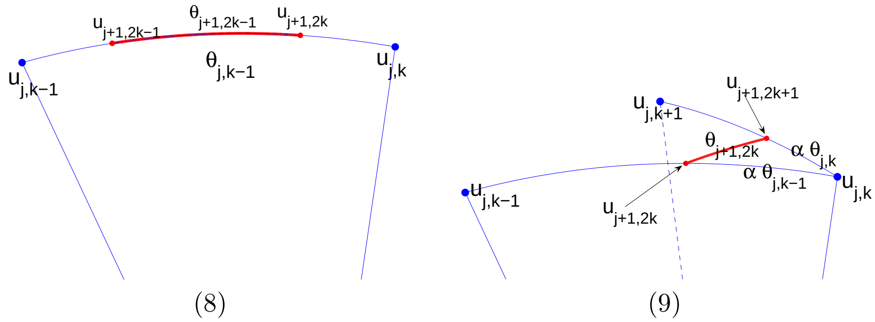

Fig. 1. Angle shrinking properties

is $s_\infty(S_\alpha)$ (any "generic data" would do), then it is clear that f_{u_0} has the same critical regularity.[1] This gives $s_\infty(T_\alpha) \leq s_\infty(S_\alpha)$, as desired. ■

Remarks. Notice the effect of the square in (16). An estimate of this form is the key "proximity condition" in the recent paper [14], which is used to prove C^1 of subdivision schemes on manifold. See also the preprint [13]. Notice that we did not use in Section any result from [14] and the proximity condition is derived 'from scratch'. The main revelation here, though, is that by combining the proximity condition (16) with the perturbation argument in [1], one obtains a result sharper than that based on the arguments in [14] alone. The author believes that a more general result can be obtained based on this idea. See also [17].

§3. Comparison with Noakes' results

After completing the writing of Section , the author became aware of the earlier work of L. Noakes [8, 7, 9]. At first glance, a big part of Theorem 2 looks just like a special case of the results in [7, 9], so this paper might be deemed almost unpublishable. At a closer look, the author realized that Theorem 2 deals with a corner cutting scheme different from the one Noakes considered in his articles: Noakes' articles study the generalization of Bézeir curves to manifold, while we deal with a parallel generalization for B-Splines. At an even closer look, the author was surprised that the two sets of results give almost totally opposite messages!

The corner-cutting schemes considered by Noakes are the ones associated with the recursive generation of a degree k polynomial represented in Bernstein form $p(t) = \sum_{\ell=0}^{k} c_\ell \binom{k}{\ell} t^\ell (1-t)^{k-\ell}$, where the iteration begins with the Bernstein coefficients $(c_\ell)_{\ell=0}^{k}$ (a.k.a. the Bézeir control

[1] Caveat: It is the parametrization f_{u_0} of the curve, not the curve itself (which is just a circle or a circular arc and is infinitely smooth), that has a finite critical Hölder exponent.

polygon in the CAGD literature.) We refer to this as the Bézeir corner cutting scheme of degree k.

(Both the Bézeir corner cutting scheme and the subdivision scheme associated with degree k B-splines ($S_{1/4}$ in Section being the latter with $k = 2$) are derived in the 1980 paper by Lane and Riesenfeld [6]; of course such recursive algorithms traced back even earlier to the classical work of de Casteljau and de Boor. For an authoritative review of such algorithms, consult Farin's book [5].)

For degree $k = 2, 3$, Noakes studied the behavior of the nonlinear scheme constructed from adapting the Bézeir corner cutting scheme to a general Riemannian manifold; the adaptation is based on the idea of replacing midpoints by geodesic midpoints. When the manifold is a sphere, we refer to this scheme as the spherical Bézeir corner cutting scheme of degree k.

We refer to $T_{1/4}$ in Section as the spherical B-spline subdivision scheme of degree 2. There are at least two different ways to apply the subdivision scheme associated with degree k B-spline to the sphere, one is based on the 'Log-Exp linearization' proposed in [11], another is based on factorizing the degree k B-spline mask into a series of two-point weighted averages, and then replacing these weighted averages by the corresponding geodesic weighted averages $\text{g-av}_\alpha(\cdot, \cdot)$ on the sphere [14]. These two approaches are the same when $k = 2$ (and both give $T_{1/4}$ in Section), but they are quantitatively different when $k > 2$. However, computational experiments suggest that, for $k = 3, 4$, the critical regularity of either one of these degree k spherical B-spline subdivision schemes is k, the same as that of a regular B-spline of degree k.

While a quadratic polynomial is infinitely smooth, a 'spherical Bézeir quadratic', i.e. the limit of the $k = 2$ geodesic corner-cutting scheme applied to 3 initial points on a sphere (see Figure 2) typically does *not* have a Hölder regularity higher than 2. This is what we observe computationally in Figure 2 below, and is actually part of the conclusion of [9]. The same behavior is observed for $k = 3, 4$, see again Figure 2. A natural question arises: what is the critical Hölder regularity of a generic curve $f : [0, 1] \to S^{n-1}$ generated by the spherical Bézeir corner cutting scheme of degree k? Perhaps the answer is 2 ($\ll \infty$!) for all k.

These observations seem quite surprising when compared to our Theorem 1: in the case of spherical Bézeir corner cutting schemes, the curviness of the sphere *does* deteriorate critical smoothness.

Besides the Lane-Riesenfeld's corner cutting algorithm, two other well-known methods for generating Bézeir curves are the degree elevation and de Casteljau's algorithms; the two latter algorithms can be generalized to Riemannian manifolds using geodesics in a way very similar to how Noakes generalized Lane-Riesenfeld's algorithm. We plan to discuss these in more details elsewhere.

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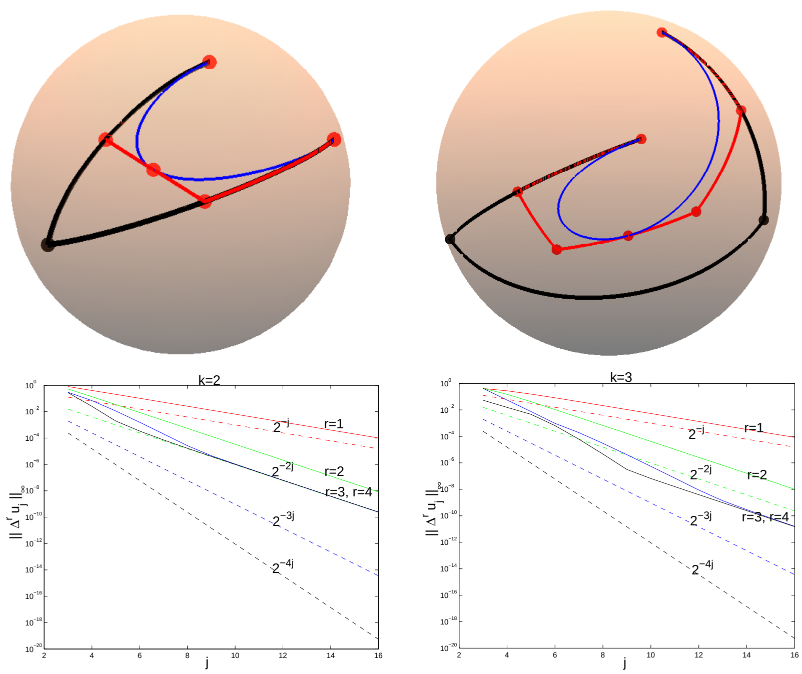

Fig. 2. Spherical Bézeir 'quadratic' (top left) & 'cubic' (top right). The control polygon of the first two iterations are also shown. In each case, the corresponding limit function $f : [0, 1] \to S^2$ is sampled on a dyadic grid of size 2^{-j}, and the infinity norms of the r-th order differences at different scales j are plotted in the log scale against j. In both cases (and many other cases not reported here), the critical Hölder smoothness of f is estimated to be 2 based on the slopes of these plots, with $r \geq 2$.