A NECESSARY AND SUFFICIENT PROXIMITY CONDITION FOR SMOOTHNESS EQUIVALENCE OF NONLINEAR SUBDIVISION SCHEMES

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Abstract. In the recent literature on subdivision methods for approximation of manifold-valued data, a certain "proximity condition" comparing a nonlinear subdivision scheme to a linear subdivision scheme has proved to be a key analytic tool for analyzing regularity properties of the scheme. This proximity condition is now well-known to be a sufficient condition for the nonlinear scheme to inherit the regularity of the corresponding linear scheme (this is called *smoothness equivalence*). Necessity, however, has remained an open problem. This paper introduces a *smooth compatibility condition* together with a new proximity condition (the *differential proximity condition*). The smooth compatibility condition makes precise the relation between non-linear and linear subdivision schemes. It is shown that under the smooth compatibility condition, the differential proximity condition is both necessary and sufficient for smoothness equivalence.

It is shown that the failure of the proximity condition corresponds to the presence of *resonance terms* in a certain discrete dynamical system derived from the nonlinear scheme. Such resonance terms are then shown to slow down the convergence rate relative to the convergence rate of the corresponding linear scheme. Finally, a *super-convergence* property of nonlinear subdivision schemes is used to conclude that the slowed decay causes a breakdown of smoothness. The proof of sufficiency relies on certain properties of the Taylor expansion of nonlinear subdivision schemes, which, in addition, explain why the differential proximity condition implies the proximity conditions that appear in previous work.

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1. Introduction

Motivated by the connection with multiscale representations of manifold-valued data and the potential impact of the approximation theory of manifold-valued data on applied areas, subdivision algorithms for manifold-valued data have been extensively studied in recent years [19, 22, 21, 27, 29,

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28, 32, 5, 12, 11, 23, 10, 24, 8, 30, 14, 13, 18]. Smoothness of the *subdivision curves* and *subdivision surfaces* produced by such algorithms is an important consideration, for smoothness is what gives curves and surfaces the smooth appearance desirable in computer graphics and engineering design.

Recall that a manifold-valued subdivision scheme S with values in a smooth manifold M is defined as follows. Let $U \subset M \times M$ be an open neighborhood of the diagonal $\Delta_M = \{(x,x) : x \in M\}$. A sequence

$$\mathbf{x} : \mathbb{Z} \to M \; : \; i \mapsto x_i$$

is said to be *sufficiently dense* if $(x_i, x_{i+1}) \in U$ for all $i \in \mathbb{Z}$. Let $\mathcal{U} \subset \ell(\mathbb{Z} \to M)$ denote the set of sufficiently dense sequences. A *(stationary) subdivision scheme* S is a map

$$S : \mathcal{U} \to \mathcal{U}\,.$$

We say that S is a C^k*-smooth* subdivision scheme, if for every sufficiently dense sequence $\mathbf{x} \in \mathcal{U}$ there is a C^k-map $F : \mathbb{R} \to M$ such that

$$\lim_{j\to\infty} (S^{j+n}\mathbf{x})_{2^{j+n}t_0} = F(t_0)\,,$$

for every dyadic integer $t_0 = k/2^n$. The map F is called the *subdivision curve* defined by $\mathbf{x}$, and $\mathbf{x}$ is called the *control data* defining F.

The problem of determining the smoothness properties of subdivision curve in this generality seems intractable. Fortunately, to the best of our knowledge, every manifold-valued subdivision scheme S appearing in the literature is based on a corresponding linear subdivision scheme, S_{lin}. Moreover, the smoothness properties of linear subdivision schemes is well understood. It is, therefore, natural to study the smoothness properties of a manifold-valued subdivision scheme S by measuring how closely it matches those of S_{lin}. This leads to the C^k*-smoothness equivalence problem*: to determine necessary and sufficient conditions for S to be C^k-smooth under the assumption that S_{lin} is C^k-smooth.

The now-standard tool for studying smoothness equivalence is the "Proximity ⇒ Smoothness Equivalence" theorem [28, Theorem 2.4], a result inspired by the work of Wallner-Dyn [22, 21] which gave sufficient conditions for S to satisfy C^1-smoothness equivalence. Similar sufficient conditions for C^k-equivalence for various special subdivision schemes have been studied extensively by Wallner [21, 3], Grohs [12, 11, 10], and by us [27, 29, 28, 32, 5, 31]. All previous work has focused on proving sufficient conditions for C^k-smoothness. The problem of giving necessary *and* sufficient conditions has remained open.

In this paper, we present a complete solution to the smoothness equivalence problem in the case where $S : \mathcal{U} \to \mathcal{U}$ is a binary subdivision scheme modeled on a stable, C^k-smooth, binary subdivision scheme S_{lin}. Our solution is based on a new proximity condition which we show is both necessary and sufficient for S to be C^k-smooth.

1.1. **The Compatibility Condition.** To state our results, we need to give a more precise definition of what it means for a manifold-valued subdivision scheme to be "modeled on" a linear subdivision scheme. Let M be a smooth manifold of dimension n without boundary and let $U \subset M \times M$ be an open neighborhood of the diagonal. For integers $L_\sigma, m_\sigma \in \mathbb{Z}$, $L_\sigma > 1$, and let

$$q_\sigma : U_{L_\sigma} \to M, \quad \sigma = 0, 1, \tag{1.1}$$

be continuous maps fixing the hyper-diagonal $M_\Delta \subset M \times \cdots \times M$, i.e.

$$q_\sigma(x, \ldots, x) = x\,, \tag{1.2}$$

where U_{L_σ} denotes the open set

$$U_{L_\sigma} = \{(x_0, x_1, \ldots, x_{L_\sigma}) \,:\, (x_i, x_{i+1}) \in U,\ \text{for } i = 0, \ldots, L_\sigma - 1\} \subset \underbrace{M \times \cdots \times M}_{L_\sigma + 1 \text{ copies}}$$

Definition 1.3. A subdivision scheme $S : \mathcal{U} \to \mathcal{U}$ is called a *binary subdivision scheme on M* if it is given by the formula

$$(S\mathbf{x})_{2i+\sigma} = q_\sigma(x_{i-m_\sigma}, \cdots, x_{i-m_\sigma+L_\sigma}), \quad \sigma = 0, 1,\ i \in \mathbb{Z}. \tag{1.4}$$

The maps q_0, q_1 are called the *even and odd rules* of S, and L_σ and m_σ are called (respectively) the *locality factors* and *phase factors* of S.

Notice that when the input sequence $\mathbf{x}$ is shifted by one entry, the subdivided sequence $S\mathbf{x}$ is shifted by two entries.

Recall that the data defining a (binary) linear subdivision scheme consist of locality and phase factors, L_σ, m_σ, together with linear functionals

$$q_{\text{lin},\sigma} : \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R} \,:\, (x_0, \ldots, x_{L_\sigma}) \mapsto \sum_{i=0}^{L_\sigma} a_{\sigma,i} x_i, \quad \sigma = 0, 1\,,$$

satisfying the sum rules $\sum_i a_{\sigma,i} = 1$. Notice that $q_{\text{lin},\sigma}$ extend to linear maps

$$q_{\text{lin},\sigma} : V \times V \times \cdots \times V \to V \,:\, (v_0, \ldots, v_{L_\sigma}) \mapsto \sum_{i=0}^{L_\sigma} a_{\sigma,i} v_i\,,$$

where V denotes any vector space over $\mathbb{R}$. The sum rules imply that $q_{\text{lin},\sigma}$ satisfies the condition (1.2), and formula (1.4) defines a subdivision scheme S_{lin} on $M = V$ for any vector space V.

Definition 1.5. We say that a subdivision scheme S is *smoothly compatible*[1] *with the linear scheme* S_{lin} if S_{lin} and S have the same phase and locality factors, and the maps q_σ are at least C^1-smooth with derivative $dq_\sigma|_{(x,\ldots,x)} : T_xM \times \cdots \times T_xM \to T_xM$ satisfying the identity

$$dq_\sigma|_{(x,\ldots,x)}(X_0, \ldots, X_{L_\sigma}) = q_{\text{lin},\sigma}(X_0, \ldots, X_{L_\sigma}), \quad \sigma = 0, 1. \tag{1.6}$$

for all $x \in M$.

Notice that the compatibility condition is coordinate independent, and, therefore, it uniquely determines the linear scheme S_{lin}. We shall, henceforth, assume that S_{lin} is a stable, C^k-smooth, binary subdivision scheme.

Remark 1.7. Our compatibility condition in Definition 1.5 is satisfied by all of the manifold-valued data subdivision schemes seen in the literature [19, 27, 29, 28, 32, 21, 12, 11, 10, 23].

[1] In [13, Definition 3.5], Grohs gives a similar compatibility condition, which he calls a "differential proximity condition."

We can encode both of the maps q_σ into a single map Q as follows. First notice that (1.1)-(1.4) imply that there is a smallest positive integer $K_{\min}$ such that any $K_{\min}+1$ consecutive entries in any (sufficiently dense) sequence $\mathbf{x}$ determines *exactly* $K_{\min}+1$ consecutive entries in $S\mathbf{x}$. We then say that S has a *minimal invariant neighborhood of size* $K_{\min}+1$. Then for any integer $K \geq K_{\min}$, the map

$$Q_K : U_K \to U_K\,, \tag{1.8}$$

is then defined as follows: For $\mathbf{x} \in \mathcal{U} \subset \ell(\mathbb{Z} \to M)$ and $\mathbf{y} = S\mathbf{x}$,

$$Q_K(x_i, \ldots, x_{i+K}) = (y_{2i+s}, \ldots, y_{2i+s+K}), \quad \text{for all } i\,. \tag{1.9}$$

Here $U_K \subset \underbrace{M \times \cdots \times M}_{K+1 \text{ copies}}$ denotes the open set of sufficiently dense $(K+1)$-tuples of points in M. The *shift factor* s is a constant independent of i but dependent on the phase factors of S, and is not uniquely determined when $K > K_{\min}$. (When $K = K_{\min}$, s, and hence also $Q_{K_{\min}}$, is uniquely determined; in general there are $2(K-K_{\min})$ choices of s. The choice of s does not matter in order for our main result to go through. See also Remark 1.20.) Since K remains fixed throughout the paper, to avoid notational clutter, we drop the subscript (i.e. $Q = Q_K$).

Remark 1.10. It is well-known that for any linear C^k subdivision scheme, the inequality $K_{\min} \geq k$ holds, with equality attained by the C^k, degree $k+1$, B-spline subdivision scheme (see Figure 1). As we shall see, the fact that K can be as small as k complicates the analysis a great deal.

$k = 1 = K$ $\qquad$ $k = 2 = K$ $\qquad$ $k = 3 = K$

$$Q(x_0,x_1) = \begin{bmatrix} q_1(x_0,x_1) \\ q_0(x_0,x_1) \end{bmatrix} \quad Q(x_0,x_1,x_2) = \begin{bmatrix} q_1(x_0,x_1) \\ q_0(x_0,x_1,x_2) \\ q_1(x_1,x_2) \end{bmatrix} \quad Q(x_0,x_1,x_2,x_3) = \begin{bmatrix} q_0(x_0,x_1,x_2) \\ q_1(x_0,x_1,x_2) \\ q_0(x_1,x_2,x_3) \\ q_1(x_1,x_2,x_3) \end{bmatrix}$$

FIGURE 1. If S is the symmetric C^k (degree $k+1$) B-Spline subdivision scheme, the corresponding map Q has a minimal invariant neighborhood of size $K+1 = k+1$. The figure shows two subdivision steps starting from $k+1$ entries of the initial sequence. (Dashes (intervals) and dots (points) are used to distinguish between the so-called primal and dual symmetries in the B-Spline subdivision schemes for odd and even k. While the two types of symmetry play no role in this paper, they play an important role in our previous studies [32, 5].)

Remark 1.11. Observe that Equation (1.2) is equivalent to the condition that Q fixes the hyper-diagonal $M_\Delta \subset M \times \cdots \times M$. Observe also that the compatibility condition (1.6) is equivalent to the condition

$$dQ|_{(x,\ldots,x)} = Q_{\mathrm{lin},x}, \quad \text{for all } x \in M, \tag{1.12}$$

where $Q_{\mathrm{lin},x} : T_xM \times \cdots \times T_xM \to T_xM \times \cdots \times T_xM$ is the corresponding linear map associated with S_{lin}.

1.2. **The Differential Proximity Condition.** Our new order k proximity condition is based on the higher order behavior of the map Q. Because it is expressed in terms of derivatives of Q, we have to impose additional smoothness assumptions on the maps q_σ. For convenience, we assume that the maps q_σ, $\sigma = 0, 1$, are infinitely differentiable.[2]

Unlike the compatibility condition, our proximity condition is most easily written in local coordinates. We first choose local coordinates for M defined on a neighborhood of an arbitrary point $p_0 \in M$ and centered so that p_0 is identified with the origin, and we now let $Q(x_0, x_1, \ldots, x_K)$ denote the local coordinate expression for Q, which is now defined on a neighborhood of the origin. In these coordinates Q fixes the hyper-diagonal $\{(x, x, \ldots, x) : x \in \mathbb{R}^n\}$ in $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$.

We next make a linear change of coordinates. Let $\nabla, \Sigma = \nabla^{-1} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the linear maps defined by the correspondence

$$(x_0, x_1, \ldots, x_K) \underset{\Sigma}{\overset{\nabla}{\rightleftarrows}} (\delta_0 = x_0, \delta_1 = x_1 - x_0, \ldots, \delta_K) \tag{1.13}$$

where $\delta_k :=$ k-th order difference of $x_0, x_1, \ldots, x_k$, so $\delta_k = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} x_\ell$, and $x_k = \sum_{\ell=0}^{k} \binom{k}{\ell} \delta_\ell$.

Finally, for $W \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{K+1 \text{ copies}}$ a sufficiently small neighborhood of the origin, define

$$\Psi : W \to \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{K+1 \text{ copies}}$$

by the formula

$$\Psi := \nabla \circ Q \circ \Sigma. \tag{1.14}$$

We write

$$\Psi = (\Psi_0, \Psi_1, \ldots, \Psi_K), \quad \Psi_\ell : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n,$$

when referring to the different components of Ψ. Observe that, in these coordinates, the fixed point set of Ψ is $\{(\delta_0, 0, \ldots, 0) : \delta_0 \in \mathbb{R}^n\} \cap W$; and the compatibility condition now assumes the form

$$d\Psi|_{(x,0,\ldots,0)} = \Psi_{\text{lin}} := \nabla \circ Q_{\text{lin}} \circ \Sigma, \qquad \text{for all } x\,. \tag{1.15}$$

We can now give a formal definition of the our new proximity condition:

Definition 1.16. Let S be a subdivision scheme on M smoothly compatible with S_{lin}. Let $k \geq 1$. We say that S satisfies the *order k differential proximity condition* if for every point $p_0 \in M$ and for local coordinates as above,

$$D^\nu \Psi_\ell|_{(\delta_0,0,\ldots,0)} = 0, \text{ when } |\nu| \geq 2, \text{ weight}(\nu) := \sum_{j=1}^{K} j\nu_j \leq \ell, \quad \text{for } 1 \leq \ell \leq k, \tag{1.17}$$

for all $(\delta_0, 0, \ldots, 0) \in W$, where ν is of the form $\nu = (0, \nu_1, \ldots, \nu_K)$.

[2]We assume that q_σ are C^∞, but our analysis only requires continuity of derivatives up to order $k+1$, where k is the order of smoothness of S_{lin}.

Notice that the proximity condition only places conditions on Ψ_ℓ for $\ell \leq k$. We shall see in Section 4, however, that the proximity condition implies the following seemingly stronger condition on *all* components of Ψ:

$$D^\nu \Psi_\ell|_{(\delta_0,0,\ldots,0)} = 0, \quad \text{when } |\nu| \geq 2, \quad \text{weight}(\nu) \leq \begin{cases} \ell, & 1 \leq \ell \leq k \\ k, & \ell > k \end{cases} . \tag{1.18}$$

The goal of this paper is to establish the following:

Theorem 1.19 (Main Result). *Let S be a subdivision scheme on a manifold smoothly compatible with the stable, C^k-smooth, linear, binary subdivision scheme S_{lin}. Then S is C^k-smooth **if and only if** it satisfies the order k differential proximity condition.*

Remark 1.20. Recall that the map Ψ depends on $Q = Q_K$, where K is any integer satisfying $K \geq K_{\min}$. Since Theorem 1.19 holds for *any* choice of $K \geq K_{\min}$, the theorem implies that the differential proximity condition is satisfied for some integer $K \geq K_{\min}$ if and only if it is satisfied for every integer $K \geq K_{\min}$. From a theoretical point of view, the case of $K = K_{\min} = k$ makes the sufficiency part most difficult to prove. In practice, we set $K = K_{\min}$, since this choice leads to maps involving the smallest number of variables.

Unlike the compatibility condition, the differential proximity condition is expressed in local coordinates. A natural question is whether the latter condition is invariant under change of coordinates. The invariance question for the original proximity condition (1.27) was answered in the affirmative in [31]. Armed with Theorem 1.19, we know that the order k differential proximity condition, being equivalent to the C^k smoothness of S, cannot be satisfied in one chart but not another, as the notion of smoothness is coordinate independent. In summary, we have:

Corollary 1.21. *If S is smoothly compatible with a C^k linear subdivision scheme S_{lin}, then the differential proximity condition of any order up to k is invariant under change of coordinates.*

1.3. **An example.** In [6] we gave an example how to apply Theorem 1.19 to obtain smoothness results for a nonlinear subdivision rule on the standard sphere in $\mathbb{R}^3$. As another example, we consider a special case of the nonlinear Lane-Riesenfeld subdivision schemes[3] studied by Dyn and Goldman in [8]. Consider the rule modeled on the degree 3 linear B-spline, consisting of data doubling, followed by two rounds of averaging using a single rule, followed by a third round of averaging alternating between two averaging rules. More specifically, in the notation of Dyn-Goldman [8], the scheme takes as input a sequence $f_i^{[0]}$ and returns as output the sequence $f_i^{[3]}$ defined by:

$$\begin{aligned} f_{2i}^{[1]} &= f_i^{[0]} & f_{2i+1}^{[1]} &= A(f_i^{[0]}, f_{i+1}^{[0]}) \\ f_{2i}^{[2]} &= A(f_{2i}^{[1]}, f_{2i+1}^{[1]}) & f_{2i+1}^{[2]} &= A(f_{2i+1}^{[1]}, f_{2i+2}^{[1]}) \\ f_{2i}^{[3]} &= A(f_{2i}^{[2]}, f_{2i+1}^{[2]}) & f_{2i+1}^{[3]} &= B(f_{2i+1}^{[2]}, f_{2i+2}^{[2]}), \end{aligned}$$

where A and B are two smooth (say C^∞) symmetric averaging rules. By the results of [8], the sequences $f^{[i]}$ converge to a C^1 function. For simplicity, assume that A is linear averaging, the

[3]We wish to thank one of the referees for pointing us to this class of nonlinear subdivision rules.

algorithm then simplifies to

$$f^{[3]}_{2i} = \frac{1}{2}(f^{[0]}_i + f^{[0]}_{i+1}) \qquad f^{[3]}_{2i+1} = B\left(\frac{1}{4}f^{[0]}_i + \frac{3}{4}f^{[0]}_{i+1}, \frac{3}{4}f^{[0]}_{i+1} + \frac{1}{4}f^{[0]}_{i+2}\right). \tag{1.22}$$

Although the general nonlinear Lane-Riesenfeld schemes considered in [8] do not fit into our framework, the above special case does. In our notation, the scheme is given by the following "even" and "odd" rules $q_0 : \mathbb{R}^3 \to \mathbb{R}$ and $q_1 : \mathbb{R}^2 \to \mathbb{R}$ given by

$$q_0(x_1, x_2, x_3) = B\left(\frac{x_1 + 3x_2}{4}, \frac{3x_2 + x_3}{4}\right) \text{ and } q_1(x_1, x_2) = \frac{x_1 + x_2}{2},$$

with $Q : \mathbb{R}^3 \to \mathbb{R}^3$ and $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$Q(x_0, x_1, x_2) = \begin{pmatrix} q_1(x_0, x_1) \\ q_0(x_0, x_1, x_2) \\ q_1(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ B\left(\frac{x_1+3x_2}{4}, \frac{3x_2+x_3}{4}\right) \\ \frac{x_1+x_2}{2} \end{pmatrix} \tag{1.23}$$

and

$$\Psi(x_0, \delta_1, \delta_2) = \begin{pmatrix} \Psi^0(x_0, \delta_1, \delta_2) \\ \Psi^1(x_0, \delta_1, \delta_2) \\ \Psi^2(x_0, \delta_1, \delta_2) \end{pmatrix} = \begin{pmatrix} x_0 + \frac{\delta_1}{2} \\ B(x_0 + \frac{3\delta_1}{4}, x_0 + \frac{5\delta_1}{4} + \frac{\delta_2}{4}) - x_0 - \frac{\delta_1}{2}) \\ -2B(x_0 + \frac{3\delta_1}{4}, x_0 + \frac{5\delta_1}{4} + \frac{\delta_2}{4}) + 2x_0 + 2\delta_1 + \frac{\delta_2}{2} \end{pmatrix}, \tag{1.24}$$

respectively, where $\delta_1 = x_1 - x_0$ and $\delta_2 = x_2 - 2x_1 + x_0$. Using the identities $B(x_0, x_0) = x_0$ and $B^{(1,0)}(x_0, x_0) = B^{(0,1)}(x_0, x_0) = 1/2$, we find that the Taylor expansion of Ψ with respect to δ_1 and δ_2 is

$$\Psi(x_0, \delta_1, \delta_2) = \begin{pmatrix} \Psi^0(x_0, \delta_1, \delta_2) \\ \Psi^1(x_0, \delta_1, \delta_2) \\ \Psi^2(x_0, \delta_1, \delta_2) \end{pmatrix} = \begin{pmatrix} x_0 \\ \frac{\delta_1}{2} \\ \frac{\delta_2}{4} + \frac{B^{(1,1)}(x_0,x_0)}{4}\delta_1^2 \end{pmatrix} + \begin{pmatrix} \text{terms of weight} >0 \\ \text{terms of weight} >1 \\ \text{terms of weight} >2 \end{pmatrix}. \tag{1.25}$$

By our main theorem, the limit functions of this scheme are C^2 if and only if $B^{(1,1)}(x_0, x_0) = 0$ for all x_0.

To give an even more concrete example, consider the following nonlinear averaging rule:

$$B(x, y) = \frac{x + y}{2} + \frac{(x - y)^p}{4}.$$

Since $B^{(1,1)}(x, x) = 0$ when $p = 4$ and $B^{(1,1)} = -1$ when $p = 2$, it follows that in the first case the subdivision rule is C^2, whereas in the second case it is only C^1. Our numerical computations in Mathematica support this result: We used control data $x = (0, 0, 0, 1, 0, 0, 0)$ to approximate the subdivision function $F(t)$ on the interval $[-2, 2]$ by applying 16 iterations of the subdivision operator S (carrying out all computations to 50 decimal places). We then estimated the first, second, and third derivatives of F using difference quotients

$$\frac{F(x + h) - F(x)}{h}, \quad \frac{F(x + h) - 2F(x) + F(x - h)}{h^2}, \quad \text{etc.}$$

with $h = 1/2^{10}$. Our results (see Figure 2) suggest (but of course does not prove) that the limit function is C^2 for $p = 4$ but only C^1 for $p = 2$.

In a separate paper, we consider the class of nonlinear degree m Lane-Riesenfeld averaging schemes where a single averaging rule is used at each round of averaging but possibility different rules are

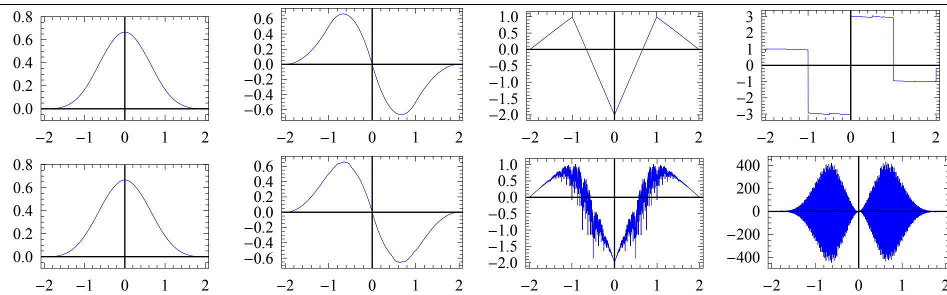

FIGURE 2. The first row above shows the graph of F, together with the graphs of its first, second, and third divided differences in the case $p = 4$. The second show shows these in the case $p = 2$.

used in $k+1$ successive rounds. We show that in this case, the order k proximity condition is satisfied and consequently the subdivision rule is C^k.

1.4. **Outline of the Proof of Theorem 1.19.** Our approach to solving the C^k-equivalence problem is to view $Q : U_K \to U_K$ as a discrete dynamical system with fixed point set the hyper-diagonal $M_\Delta \subset M \times M \cdots \times M$ and to show that the convergence properties of the sequences generated by iterating Q govern the C^k-smoothness of S.

Recall that $S_{\rm lin}$ is assumed to be a stable, C^k-smooth subdivision rule, with $k \geq 1$. It follows (see Lemma 2.2 below) that the dominant eigenvalue of $Q_{\rm lin}$ is 1 with multiplicity $\dim(M)$ and with the tangent space of the hyper-diagonal the corresponding eigenspace. In addition, the subdominant eigenvalue of $Q_{\rm lin}$ is $1/2$, from which it follows that $Q_{\rm lin}$ is contractive in directions normal to M_Δ. This in turn implies that M_Δ is a basin of attraction of the (nonlinear) map Q, i.e., there is a (possibly smaller) Q-invariant neighborhood $U \subset U_K$ of M_Δ such that every sequence

$$Q^j(x_0, x_1, \ldots, x_K), \quad j = 1, 2, \ldots, \text{ for } (x_0, x_1, \ldots . x_K) \in U$$

of iterates of Q converges to a point in M_Δ. We show that the C^k-smoothness of S is intimately connected with the convergence properties of such sequences.

To see why this is the case, we change coordinates to $(\delta_0, \ldots, \delta_K)$. Since M_Δ is a basin of attraction of Q, it follows we can find a neighborhood $V \subset W$ of the origin of the form

$$V = V_0 \times D \subset \mathbb{R}^n \times \underbrace{(\mathbb{R}^n \times \cdots \times \mathbb{R}^n)}_{K-\text{copies}}$$

such that $\Psi^j(V) \subset W$ for all j. and such that for all $\boldsymbol{\delta} = (\delta_0, \delta_1, \ldots, \delta_K) \in V$, the sequence

$$\boldsymbol{\delta}^{(j)} = (\delta_0^{(j)}, \delta_1^{(j)}, \ldots, \delta_K^{(j)}) := \Psi^j(\boldsymbol{\delta})$$

converges to a point in $\mathbb{R}^n \times (0, \ldots, 0)$ as $j \to \infty$. This implies that $\delta_\ell^{(j)} \to 0$ for $1 \leq \ell \leq K$.

Necessity. Under the assumption that S is C^k-smooth, one suspects more, since for $1 \leq \ell \leq k$ the sequence of divided differences $2^{j\ell}\delta_\ell^{(j)}$ should (in some sense) approximate the ℓ-th order derivatives of a C^k-smooth subdivision curve. In Section 5.2, we prove a "super-convergence" result (Theorem 5.48) that justifies this assumption and implies that the differences $\delta_\ell^{(j)}$ must decay at least as fast as $2^{-j\ell}$ for $\ell \le k$.

On the other hand, we show that when our order k differential proximity condition fails, the differences $\delta_k^{(j)}$ decrease slower than 2^{-jk} (Theorem 5.13). This phenomenon is due to the presence of certain "resonance terms" in the Taylor expansion of the map Ψ. These resonance terms correspond to the non-vanishing of the derivatives in (1.17).

Combining the above two remarks shows that the differential proximity condition is a necessary condition for S to be C^k.

Sufficiency. Our proof that the order k differential proximity condition is sufficient for S to be C^k proceeds by showing that the differential proximity condition implies the previously known proximity condition in [28], which is already known to be sufficient. As we shall see, our differential proximity condition appears on the surface to be weaker than the condition in [28]. That this is not the case is a consequence of our *Alternating Sign Lemma* (Lemma 4.7), which reveals a rather subtle structure in the Taylor expansion of the map Ψ.

1.5. **Relation with Previous Proximity conditions.** To set the stage, we review the statement of [28, Theorem 2.4]. In its most general form[4] the "Proximity ⇒ Smoothness Equivalence" theorem states that if S and S_{lin} satisfy the order k proximity condition, which reads

$$\|\Delta^{j-1}S\mathbf{x} - \Delta^{j-1}S_{\text{lin}}\mathbf{x}\|_\infty \le C\,\Omega_j(\mathbf{x}), \quad j = 1, \dots, k, \tag{1.26}$$

where

$$\Omega_j(\mathbf{x}) := \sum_{\gamma\in\Gamma_j} \prod_{i=1}^{j} \|\Delta^i \mathbf{x}\|_\infty^{\gamma_i}, \quad \Gamma_j := \left\{ \gamma = (\gamma_1, \cdots, \gamma_j) \,\middle|\, \gamma_i \in \mathbb{Z}^+, \sum_{i=1}^{j} i\,\gamma_i = j+1 \right\}, \tag{1.27}$$

and if S_{lin} is C^k-smooth and L_∞-stable, then S is C^k-smooth.

The above proximity condition has a number of defects:

• In the setting of [28, Theorem 2.4], the question of necessity involves the existence of an unspecified linear subdivision rule. For in this theorem, and in all the manifold-valued data subdivision schemes considered in the literature, the nonlinear scheme S is meant to be constructed from an underlying linear scheme S_{lin}; but no general methods for computing S_{lin} from S are given. Rather, one takes as data a nonlinear scheme S *together with* a linear scheme S_{lin} with known regularity, say C^k. To show that S has regularity C^k, one then verifies that the proximity conditions are satisfied by the pair S, S_{lin}.

• Even when S_{lin} is known, the proximity condition is difficult to check directly. (See, for instance, the computations in [5].)

• The underlying reasons why the proximity condition implies smoothness are unclear.

[4]Other authors explore generalizations to settings where the domain space is multi-dimensional but for low order smoothness, see for example [9, 25].

• There is a more perplexing (and in fact rather embarrassing) problem with the proximity condition (1.26): it appears to be unnecessarily strong. This phenomenon already appears when $k = 1$, where the proximity condition (1.26) assumes the form

$$\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty \leq C\|\Delta\mathbf{x}\|_\infty^2 \,. \tag{1.28}$$

Condition (1.28) was first proposed by Wallner and Dyn in [21] and shown by them to be a sufficient condition for both C^0 and C^1 equivalence. In [28, Theorem 2.3], it is shown that if the order 1 proximity condition in (1.26) is satisfied, then C^0 regularity in S follows, assuming that S_{lin} also has C^0 regularity; but upon close inspection of the proof of this result, one sees that (1.28) can be replaced by the weaker condition:

$$\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty \leq C\|\Delta\mathbf{x}\|_\infty^{1+\epsilon}, \;\; \epsilon > 0. \tag{1.29}$$

Thus, it appears that while (1.28) is a convenient single condition for inferring C^0 and C^1 equivalence simultaneously, we could first prove the weaker condition (1.29) for C^0 equivalence; and the following *weaker* $k = 1$ proximity condition

$$\|\Delta S\mathbf{x} - \Delta S_{\text{lin}}\mathbf{x}\|_\infty \leq C\|\Delta\mathbf{x}\|_\infty^2 \tag{1.30}$$

would be sufficient to infer C^1 equivalence.

The problem persists for $k > 1$, for if one carefully inspects the proof of [28, Theorem 2.4], one sees that the weaker proximity condition

$$\|\Delta^j S\mathbf{x} - \Delta^j S_{\text{lin}}\mathbf{x}\|_\infty \leq C\,\Omega_j(\mathbf{x}), \quad j = 1, \ldots, k, \tag{1.31}$$

is all that is needed to prove sufficiency provided that the C^0 regularity of S is already established. (See [26] for the details.) Henceforth, we shall refer to (1.26) as the *strong proximity condition* of order k and to (1.31) as the *weak proximity condition* of order k. To see that the strong proximity condition implies the weak proximity condition, estimate as follows:

$$\|\Delta^j S\mathbf{x} - \Delta^j S_{\text{lin}}\|_\infty = \|\Delta(\Delta^{j-1} S\mathbf{x} - \Delta^{j-1} S_{\text{lin}}\mathbf{x})\|_\infty \leq 2\|\Delta^{j-1} S\mathbf{x} - \Delta^{j-1} S_{\text{lin}}\mathbf{x}\|_\infty \leq C\Omega_j(\mathbf{x})\,.$$

On the other hand, while the proximity condition (1.26) appears too strong, there is ample numerical evidence (e.g. see [32]) suggesting that this condition is also *necessary.* We are thus faced with an apparent contradiction.

Our *differential proximity condition* resolves all of these problems. It is not only easy to verify, but it also has a relatively clear interpretation in terms of dynamical systems. Finally, as Figure 3 illustrates, the new theory resolves the apparent contradiction described above by showing that all three order k proximity conditions are equivalent and necessary and sufficient for C^k-equivalence.

In particular, the subdivision schemes presented in [28] and [5] where the strong proximity condition is violated are not C^k-schemes.

1.6. **Organization of the Paper.** The remainder of the paper is organized as follows. We discuss in Section 2 various properties of linear subdivision schemes that we use throughout the paper. In Section 3 we prove two technical decay results about the sequence $\boldsymbol{\delta}^{(j)}$ discussed above. Sections 4 and 5 contain the proofs of sufficiency and necessity in Theorem 1.19.

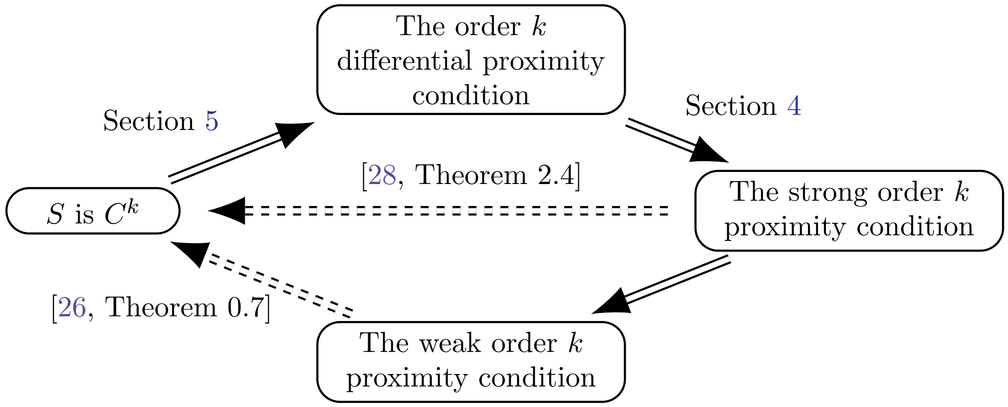

FIGURE 3. The network of implcations when S is compatible with a stable, C^k subdivision rule S_{lin}. Implications proved in this paper are indicated by solid arrows; those proved elsewhere are indicated by dashed arrows.

1.7. **Notation.** Henceforth, we shall work in local coordinates. For simplicity, we abuse notation and let $\mathbf{x}$ denote a sufficiently dense, sometimes finite and sometimes doubly-infinite sequence in $\mathbb{R}^n$. The meaning will always be clear from context.

We let $\Delta^\ell \mathbf{x}$ denote the sequence of ℓ-th order finite differences defined by $\mathbf{x}$. When $\mathbf{x}$ is a finite instead of a bi-infinite sequence, the error terms $\Omega_\ell(\mathbf{x})$, $j = 1, 2, \ldots$, are defined as in (1.27), except that now $\mathbf{x}$ is a finite sequence of length $K + 1$ and so

$$\|\Delta^i \mathbf{x}\|_\infty := \max_{r=0,\ldots,K-i} \|(\Delta^i \mathbf{x})_r\|_\infty.$$

By convention, $\Omega_0(\mathbf{x}) := 0$.

Finally, we use the following notation for iterates:

$$Q^j \mathbf{x} = \underbrace{Q \circ Q \circ \cdots \circ Q}_{j\text{-times}}(x_0, x_1, \ldots, x_K) \text{ and } \Psi^j \boldsymbol{\delta} = \underbrace{\Psi \circ \Psi \circ \cdots \circ \Psi}_{j\text{-times}}(\delta_0, \delta_1, \ldots, \delta_K)\,,$$

for $\mathbf{x} = (x_0, x_1, \ldots, x_K)$ and $\boldsymbol{\delta} = (\delta_0, \delta_1, \ldots, \delta_K)$ sufficiently dense finite sequences of points in $\mathbb{R}^n$.

2. Some Properties of Linear Subdivision Rules

In this section, we discuss properties of linear subdivision rules that play a role in our proof of Theorem 1.19.

Recall from the linear theory that when S_{lin} is C^k and stable, it must also reproduce Π_k (the space of polynomials of degree not exceeding k) and, more specifically,

$$\forall\, p \in \Pi_k, \exists\, q \in \Pi_k \text{ such that } S_{\text{lin}}(p|_{\mathbb{Z}}) = q|_{\frac{1}{2}\mathbb{Z}} \text{ and } \deg(p - q) < \deg(p). \tag{2.1}$$

In other words, S_{lin} maps every degree k polynomial sequence to some polynomial sequence with the same leading monic term. Under these conditions, there exists also a so-called derived subdivision scheme $S_{\rm lin}^{[\ell]}$ such that $\Delta^\ell \circ S_{\rm lin} = S_{\rm lin}^{[\ell]} \circ \Delta^\ell$ for every $\ell = 1, \ldots, k+1$. See [7, 1, 20, 2] for details on these results.

The following lemma shows that when $S_{\rm lin}$ is C^k and stable, the linear map $\Psi_{\rm lin}$ is upper triangular, with eigenvalues $2^{-\ell}$ arranged in decreasing order along the diagonal, and with the off-diagonal terms in the ℓ-th row of weight greater than ℓ. The differential proximity condition can be viewed as a nonlinear version of this same result.

Lemma 2.2. *If $S_{\rm lin}$ satisfies (2.1), then $\Psi_{\rm lin}$ has the block* ***upper triangular*** *form:*

$$\Psi_{{\rm lin},\ell}(\delta_0, \delta_1, \ldots, \delta_K) = \begin{cases} \frac{1}{2^\ell}\delta_\ell + \sum_{\ell'=\ell+1}^K U_{\ell,\ell'}\delta_{\ell'}, & \ell = 0, \ldots, k \\ \sum_{\ell'=k+1}^K U_{\ell,\ell'}\delta_{\ell'}, & \ell = k+1, \ldots, K \end{cases}, \tag{2.3}$$

where $U_{\ell,\ell'}$ are scalars *dependent only on the mask of $S_{\rm lin}$. Moreover, if $S_{\rm lin}$ is C^k smooth, the spectral radius of the lower right block $[U_{\ell,\ell'}]_{\ell,\ell'=k+1,\ldots,K}$ is strictly smaller than $1/2^k$.*

Proof. To simplify notation, assume that the dimension of the manifold, n, is 1. Extending the argument to $n > 1$ is trivial, as the compatibility condition says that the linear part of S is the linear scheme $S_{\rm lin}$ applied *component-wise*. By (2.1), if we define the (Vandermonde) matrix $[P_{\ell,\ell'}]_{0\le\ell,\ell'\le K}$, $P_{\ell,\ell'} = \ell^{\ell'}$, then $Q_{\rm lin}P = P\hat{U}$, where $\hat{U}$ has exactly the same structure as the block upper triangular matrix U as claimed above. (2.3) then follows from the similarity relation between $Q_{\rm lin}$ and $\Psi_{\rm lin}$ through (1.15) and that an order k difference operator annihilates polynomials of degree less than k, we omit the routine linear algebra derivation. The second part of the lemma follows from the fact that a linear C^k subdivision scheme must have some "excess smoothness" $C^{k,\alpha}$, $\alpha > 0$, so $\|\Delta^\ell S_{\rm lin}^j \mathbf{x}\|_\infty = O(2^{-j(k+\alpha)})$ when $\ell \ge k+1$, which also means that the lower right block $[U_{\ell,\ell'}]_{\ell,\ell'=k+1,\ldots,K}$ has a spectral radius strictly smaller than $1/2^k$. □

2.1. **The stability trick in linear theory.** It is well known from the linear theory that

$$\|\Delta^r S_{\rm lin}^j \mathbf{x}\|_\infty \lesssim 2^{-j(m+\alpha)}, \;\; r > m, \;\; \text{for all } \mathbf{x} \Longrightarrow S_{\rm lin} \text{ is } C^{m,\alpha}. \tag{2.4}$$

It is known that the converse is true only if we assume additionally that $S_{\rm lin}$ is stable, i.e. if $S_{\rm lin}$ is stable then

$$S_{\rm lin} \text{ is } C^{m,\alpha} \iff \|\Delta^r S_{\rm lin}^j \mathbf{x}\|_\infty \lesssim 2^{-j(m+\alpha)}, \text{ for all } \mathbf{x}, \;\; r > m. \tag{2.5}$$

See, for example, [1, 20]

Recall the definition of stability. We assume that $S_{\rm lin}$ has a finitely supported mask, and hence a compactly supported refinable function ϕ. Then $S_{\rm lin}$ or ϕ is called (L^∞-)*stable* if

$$A\|\mathbf{x}\|_{\ell^\infty} \le \Big\| \sum_{k\in\mathbb{Z}} \mathbf{x}_k \phi(t-k) \Big\|_{L^\infty} \le B\|\mathbf{x}\|_{\ell^\infty} \tag{2.6}$$

for some constants $A, B > 0$. Since ϕ is bounded and compactly supported, the upper bound above is automatic. It is the lower bound that can fail for some subdivision schemes.

The trick for proving (2.5) is as follows. Let $f(t) = \sum_k x_k \phi(t-k)$; assume that ϕ, and therefore f, is $C^{m,\alpha}$ smooth and that ϕ satisfies the stability condition (2.6). Then $\left\|\Delta^r_{2^{-j}} f\right\|_\infty = O(2^{-j(m+\alpha)})$.

By the connection between subdivision and refinability,

$$f(t) = \sum_k x_k \phi(x-k) = \sum_k (S^j_{\text{lin}} x)_k \phi(2^j t - k).$$

Consequently, $\Delta^r_{2^{-j}} f(t) = \Delta^r_{2^{-j}} \sum_k (S^j_{\text{lin}} x)_k \phi(2^j t - k) = \sum_k \left(\Delta^r S^j_{\text{lin}} \mathbf{x}\right)_k \phi(2^j t - k)$. Then, by stability,

$$\|\Delta^r S^j_{\text{lin}} \mathbf{x}\|_{\ell^\infty} \lesssim \|\Delta^r_{2^{-j}} f\|_{L^\infty} = O(2^{-j(m+\alpha)}).$$

Notice that both the stability condition itself and the argument above rely heavily on linearity.

3. Decay Results

In this section, we prove two technical results about linear subdivision schemes and the asymptotic behavior of the iterates $Q^j(\mathbf{x})$ and $\Psi^j(\boldsymbol{\delta})$ as $j \to \infty$, which we need for both the necessity and sufficiency parts of the proof of Theorem 1.19.

Proposition 3.1. *(a) Suppose that there exists a constant $C > 0$ such the following conditions are satisfied for every finite sequence $\mathbf{x} = (x_0, \dots, x_K)$ of points in $\mathbb{R}^n$:*

$$\|\Delta^\ell Q^j_{\text{lin}} \mathbf{x}\|_\infty \le C \|\Delta^\ell \mathbf{x}\|_\infty 2^{-\ell j}, \quad \ell = 1, \cdots, s, \tag{3.2a}$$

$$\|\Delta^{s+1} Q^j_{\text{lin}} \mathbf{x}\|_\infty \le C \|\Delta^{s+1} \mathbf{x}\|_\infty 2^{-j(s+\alpha)}, \quad \alpha \in (0, 1], \tag{3.2b}$$

and

$$\|\Delta^\ell Q\mathbf{x} - \Delta^\ell Q_{\text{lin}} \mathbf{x}\|_\infty \le C \Omega_\ell(\mathbf{x}), \quad \ell = 1, \cdots, s, . \tag{3.2c}$$

Then for every $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$\|\Delta^\ell Q^j \mathbf{x}\|_\infty \le C_\epsilon 2^{-j(\ell-\epsilon)} (\Omega_{\ell-1}(\mathbf{x}) + \|\Delta^\ell \mathbf{x}\|_\infty), \quad \ell = 1, \cdots, s, \tag{3.3a}$$

and

$$\|\Delta^{s+1} Q^j \mathbf{x}\|_\infty \le C_\epsilon 2^{-j(s+\alpha-\epsilon)} (\Omega_s(\mathbf{x}) + \|\Delta^{s+1} \mathbf{x}\|_\infty). \tag{3.3b}$$

for every sufficiently dense sequence $\mathbf{x}$.

(b) Suppose that there exists a constant $C > 0$ such that the following conditions are satisfied for every doubly infinite sequence $\mathbf{x}$ of points in $\mathbb{R}^n$:

$$\|\Delta^\ell S^j_{\text{lin}} \mathbf{x}\|_{\ell^\infty} \le C \|\Delta^\ell \mathbf{x}\|_{\ell^\infty} 2^{-\ell j}, \quad \ell = 1, \cdots, s, \tag{3.4a}$$

$$\|\Delta^{s+1} S^j_{\text{lin}} \mathbf{x}\|_{\ell^\infty} \le C \|\Delta^{s+1} \mathbf{x}\|_{\ell^\infty} 2^{-j(s+\alpha)}, \quad \alpha \in (0, 1], \tag{3.4b}$$

$$\|\Delta^\ell S\mathbf{x} - \Delta^\ell S_{\text{lin}} \mathbf{x}\|_\infty \le C\, \Omega_\ell(\mathbf{x}), \quad \ell = 1, \dots, s. \tag{3.4c}$$

(Note that (3.4c) is the weak proximity condition of order s.) Then for all $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such

$$\|\Delta^\ell S^j \mathbf{x}\|_{\ell^\infty} \le C_\epsilon 2^{-j(\ell-\epsilon)} (\Omega_{\ell-1}(\mathbf{x}) + \|\Delta^\ell \mathbf{x}\|_\infty), \quad \ell = 1, \cdots, s. \tag{3.5}$$

and

$$\|\Delta^{s+1} S^j \mathbf{x}\|_{\ell^\infty} \le C_\epsilon 2^{-j(s+\alpha-\epsilon)} (\Omega_s(\mathbf{x}) + \|\Delta^{s+1} \mathbf{x}\|_\infty). \tag{3.6}$$

for every sufficiently dense doubly infinite sequence $\mathbf{x}$.

Proof. We first prove part (a). The proof proceed in four steps:

Step (i). It follows from (3.2a) and (3.2b) that for arbitrarily small $\epsilon_1, \cdots, \epsilon_s, \epsilon_{s+1} > 0$, we can choose a big enough power $m = m(\epsilon_1, \ldots, \epsilon_{s+1})$ such that

$$\|\Delta^\ell Q^m_{\rm lin}\mathbf{x}\|_\infty \le \|\Delta^\ell \mathbf{x}\|_\infty 2^{-m(\ell-\epsilon_\ell)}, \quad \ell = 1, \cdots, s, \tag{3.7}$$

$$\|\Delta^{s+1} Q^m_{\rm lin}\mathbf{x}\|_\infty \le \|\Delta^{s+1} \mathbf{x}\|_\infty 2^{-m(s+\alpha-\epsilon_{s+1})}. \tag{3.8}$$

This establishes "one-step decay rates" pertaining to a "powered" version of the linear map $Q_{\rm lin}$.

In step (ii), we establish a similar one-step decay rate for Q raised to the same power (m). In step (iii), we use (i) and (ii) and the proximity condition to establish a family of asymptotic decay rates of Q^m. Step (iii) essentially proves the proposition, only with Q replaced by Q^m. In Step (iv), we reduce the power m back to unity by sacrificing the size of a hidden constant.

Step (ii). It can be shown that (see [28, Lemma A.3]) when the proximity condition (3.2c) holds then the same proximity condition holds between Q^m and $Q^m_{\rm lin}$ for any power $m \in \mathbb{N}$, i.e. there exists $C_m > 0$ such that

$$\|\Delta^\ell Q^m\mathbf{x} - \Delta^\ell Q^m_{\rm lin}\mathbf{x}\|_\infty \le C_m \Omega_\ell(\mathbf{x}), \quad \ell = 1, \cdots, s. \tag{3.9}$$

In particular, by applying this to the case of $\ell = 1$ and the power $m = m(\epsilon_1)$ from step (i), we have:

$$\|\Delta Q^m\mathbf{x}\|_\infty \le \|\Delta Q^m_{\rm lin}\mathbf{x}\|_\infty + C_m\|\Delta\mathbf{x}\|^2_\infty.$$

It then follows from (3.7) that

$$\|\Delta Q^m\mathbf{x}\|_\infty \le \|\Delta\mathbf{x}\|_\infty 2^{-m(1-\epsilon_1)} + C_m\|\Delta\mathbf{x}\|^2_\infty = \|\Delta\mathbf{x}\|_\infty(2^{-m(1-\epsilon_1)} + C_m\|\Delta\mathbf{x}\|_\infty).$$

Therefore, for any

$$\bar\epsilon_1 > \epsilon_1, \tag{3.10}$$

we can choose $\mathbf{x}$ dense enough such that

$$2^{-m(1-\epsilon_1)} + C_m\|\Delta\mathbf{x}\|_\infty \le 2^{-m(1-\bar\epsilon_1)}.$$

Hence, there is a δ which depends on $\bar\epsilon_1, \epsilon_1$ so that

$$\|\Delta Q^m\mathbf{x}\|_\infty \le \|\Delta\mathbf{x}\|_\infty 2^{-m(1-\bar\epsilon_1)}, \quad \text{for } \|\Delta\mathbf{x}\|_\infty < \delta := \delta(\bar\epsilon_1, \epsilon_1). \tag{3.11}$$

Step (iii). In this step, we use (i) and (ii) to prove the following claim by *induction on the differencing order* ℓ:

For any $\bar\epsilon_1, \epsilon_1 > 0$ that satisfy (3.10) and $\bar\epsilon_2, \cdots, \bar\epsilon_s, \bar\epsilon_{s+1} > 0$ that satisfy

$$\bar\epsilon_{\ell+1} > (\ell+1)\bar\epsilon_\ell, \quad \ell = 1, \cdots, s, \tag{3.12}$$

there exist $B_1, \cdots, B_s, B_{s+1} > 0$ such that

$$\|\Delta^\ell Q^{mj}\mathbf{x}\|_\infty \le B_\ell\left(\Omega_{\ell-1}(\mathbf{x}) + \|\Delta^\ell\mathbf{x}\|_\infty\right) 2^{-mj(\ell-\bar\epsilon_\ell)}, \quad \ell = 1, \cdots, s \tag{3.13}$$

$$\|\Delta^{s+1} Q^{mj}\mathbf{x}\|_\infty \le B_{s+1}\left(\Omega_s(\mathbf{x}) + \|\Delta^{s+1}\mathbf{x}\|_\infty\right) 2^{-mj(s+\alpha-\bar\epsilon_{s+1})} \tag{3.14}$$

hold for $m = m(\epsilon_1, \bar\epsilon_2, \ldots, \bar\epsilon_{s+1})$ (established in Step (i)) and sequences $\mathbf{x}$ that satisfy $\|\Delta\mathbf{x}\|_\infty < \delta(\bar\epsilon_1, \epsilon_1)$ (established in Step (ii).)

First, note that (3.13) with $\ell = 1$ follows from iterating the "one-step decay" established in part (ii). We now proceed by induction on ℓ. For this purpose, assume that (3.13) holds with $\ell \leq L$ for some $L \leq s$ and seek to prove either (3.13) with $\ell = L + 1$ (if $L < s$) or (3.14) (if $L = s$.)

Using the "power proximity condition" (3.9):

$$\begin{aligned}\|\Delta^{L+1}Q^{mj}\mathbf{x}\|_\infty &\leq \|\Delta^{L+1}Q^{mj}\mathbf{x} - \Delta^{L+1}Q^m_{\text{lin}}Q^{m(j-1)}\mathbf{x}\|_\infty + \|\Delta^{L+1}Q^m_{\text{lin}}Q^{m(j-1)}\mathbf{x}\|_\infty \\ &\leq 2\|\Delta^{L}Q^{mj}\mathbf{x} - \Delta^{L}Q^m_{\text{lin}}Q^{m(j-1)}\mathbf{x}\|_\infty + \|\Delta^{L+1}Q^m_{\text{lin}}Q^{m(j-1)}\mathbf{x}\|_\infty \\ &\leq 2C_m\Omega_L(Q^{m(j-1)}\mathbf{x}) + \|\Delta^{L+1}Q^m_{\text{lin}}Q^{m(j-1)}\mathbf{x}\|_\infty.\end{aligned} \tag{3.15}$$

It follows from the induction hypothesis and the definitions of γ_L and $\Omega_L(\mathbf{x})$ that

$$\begin{aligned}\Omega_L(Q^{mi}\mathbf{x}) &= \sum_{\gamma\in\Gamma_L}\prod_{\ell=1}^{L}\|\Delta^\ell Q^{mi}\mathbf{x}\|_\infty^{\gamma_\ell} \\ &\leq \sum_{\gamma\in\Gamma_L}\prod_{\ell=1}^{L}B_\ell^{\gamma_\ell}(\Omega_{\ell-1}(\mathbf{x}) + \|\Delta^\ell\mathbf{x}\|_\infty)^{\gamma_\ell}2^{-mi(\ell-\bar{\epsilon}_\ell)\gamma_\ell} \\ &\leq \underbrace{\max_{\gamma\in\Gamma_L}\prod_{\ell=1}^{L}B_\ell^{\gamma_\ell}}_{=O(1)}\sum_{\gamma\in\Gamma_L}\underbrace{\prod_{\ell=1}^{L}(\Omega_{\ell-1}(\mathbf{x}) + \|\Delta^\ell\mathbf{x}\|_\infty)^{\gamma_\ell}}_{=O(\Omega_L(\mathbf{x}))}\underbrace{\prod_{\ell=1}^{L}2^{-mi(\ell-\bar{\epsilon}_\ell)\gamma_\ell}}_{=2^{-mi(L+1-\sum_{\ell=1}^{L}\bar{\epsilon}_\ell\gamma_\ell)}}.\end{aligned} \tag{3.16}$$

The assumption (3.12) means in particular that $\bar{\epsilon}_L > \bar{\epsilon}_{L-1} > \cdots > \bar{\epsilon}_1$. Hence, $\sum_{\ell=1}^{L}\bar{\epsilon}_\ell\gamma_\ell \leq \bar{\epsilon}_L\sum_{\ell=1}^{L}\gamma_\ell \leq \bar{\epsilon}_L\sum_{\ell=1}^{L}\ell\gamma_\ell = \bar{\epsilon}_L(L+1)$.

Combining this with (3.16), we have

$$\Omega_L(Q^{mi}\mathbf{x}) \leq A\Omega_L(\mathbf{x})2^{-mi(L+1)(1-\bar{\epsilon}_L)}, \tag{3.17}$$

for some constant $A > 0$.

It follows from the "one-step decay" estimate (3.7)-(3.8) that

$$\|\Delta^{L+1}Q^m_{\text{lin}}Q^{m(j-1)}\mathbf{x}\|_\infty \leq \begin{cases}\|\Delta^{L+1}Q^{m(j-1)}\mathbf{x}\|_\infty 2^{-m(L+1-\bar{\epsilon}_{L+1})} & \text{if } L < s \\ \|\Delta^{L+1}Q^{m(j-1)}\mathbf{x}\|_\infty 2^{-m(s+\alpha-\bar{\epsilon}_{s+1})} & \text{if } L = s\end{cases}. \tag{3.18}$$

We first deal with the case of $L < s$. For convenience, we write $\rho := 2^{-m(L+1-\bar{\epsilon}_{L+1})}$, $\tilde{\rho} := 2^{-m(L+1)(1-\bar{\epsilon}_L)}$.

Now, using (3.15), (3.17) and (3.18), we have:[5]

$$
\begin{aligned}
\|\Delta^{L+1}Q^{mj}\mathbf{x}\|_\infty &\overset{(3.15)+(3.18)}{\leq} \rho\|\Delta^{L+1}Q^{m(j-1)}\mathbf{x}\|_\infty + 2C_m\Omega_L(Q^{m(j-1)}\mathbf{x}) \\
&\overset{(3.15)+(3.18)}{\leq} \rho^2\|\Delta^{L+1}Q^{m(j-2)}\mathbf{x}\|_\infty + \rho\cdot 2C_m\Omega_L(Q^{m(j-2)}\mathbf{x}) + 2C_m\Omega_L(Q^{m(j-1)}\mathbf{x}) \\
&\leq \cdots \leq \rho^j\|\Delta^{L+1}\mathbf{x}\|_\infty + 2C_m\sum_{i=0}^{j-1}\rho^{j-i-1}\Omega_L(Q^{mi}\mathbf{x}) \\
&= \rho^j\left[\|\Delta^{L+1}\mathbf{x}\|_\infty + 2C_m\rho^{-1}\sum_{i=0}^{j-1}\rho^{-i}\Omega_L(Q^{mi}\mathbf{x})\right] \\
&\overset{(3.17)}{\leq} \rho^j\left[\|\Delta^{L+1}\mathbf{x}\|_\infty + 2C_m\rho^{-1}A\Omega_L(\mathbf{x})\sum_{i=0}^{j-1}\rho^{-i}\tilde{\rho}^i\right].
\end{aligned}
$$

To finish the proof of this part, we just have to check that the sum $\sum_{i=0}^{j-1}(\tilde{\rho}/\rho)^i$ does not blow up when j grows; but this is indeed the case, as we tune the epsilon's to satisfy (3.12) so that $\tilde{\rho}/\rho = 2^{m[(L+1)\bar{\epsilon}_L - \bar{\epsilon}_{L+1}]} < 1$. Therefore, the decay estimate (3.13) holds for $\ell = L+1$ with a suitably chosen constants B_{L+1}; we are done with the induction step in the case of $L < s$.

When $L = s$, (3.18) takes the form of the second estimate, the same argument above yields (3.14) for a suitable choice of constant B_{s+1}.

Step (iv). For any $\epsilon > 0$ as in the statement of this lemma, we can choose $\epsilon_1, \bar{\epsilon}_1, \cdots, \bar{\epsilon}_s, \bar{\epsilon}_{s+1}$ as in step (iii) with $\bar{\epsilon}_{s+1} = \epsilon$, so that (3.13) and (3.14) hold for a large enough power m and all dense enough data $\mathbf{x}$.

For any integer $j \geq 0$, we can write it as $j = mq + r$, with $0 \leq r \leq m-1$. For any $\ell = 1, \cdots, s$, we have

$$
\begin{aligned}
\|\Delta^\ell Q^j\mathbf{x}\|_\infty = \|\Delta^\ell Q^{mq}Q^r\mathbf{x}\|_\infty &\leq B_\ell 2^{-mq(\ell-\epsilon)}(\Omega_{\ell-1}(Q^r\mathbf{x}) + \|\Delta^\ell Q^r\mathbf{x}\|_\infty) \\
&= \underbrace{(B_\ell 2^{r(\ell-\epsilon)})}_{\text{“hidden constant”}}\; 2^{-j(\ell-\epsilon)}\underbrace{(\Omega_{\ell-1}(Q^r\mathbf{x}) + \|\Delta^\ell Q^r\mathbf{x}\|_\infty)}_{(*)}.
\end{aligned}
\tag{3.19}
$$

It is easy to show (see below) that there exists $D_r > 0$ such that

$$
\Omega_\ell(Q^r\mathbf{x}) \leq D_r\Omega_\ell(\mathbf{x}) \quad \text{and} \quad \|\Delta^\ell Q^r\mathbf{x}\|_\infty \leq D_r(\Omega_{\ell-1}(\mathbf{x}) + \|\Delta^\ell\mathbf{x}\|_\infty), \quad \ell = 1, 2, \cdots, s. \tag{3.20}
$$

for all dense enough data. By applying (3.20) to $(*)$, we can 'trade' any $r = 1, \ldots, m-1$ in $(*)$ with $r = 0$ but a bigger "hidden constant", meaning that (3.3a) can be established with a big enough constant C_ϵ. The proof of (3.3b) is similar.

[5] Note how this step would fail if we had an unknown constant $C > 1$ in front of the right-hand side of (3.18).

To prove (3.20), we use again the power proximity condition (3.9) to get:

$$
\begin{aligned}
\|\Delta^\ell Q^r \mathbf{x}\|_\infty \leq& \|\Delta^\ell Q^r_{\rm lin}\mathbf{x}\|_\infty + C_r\Omega_\ell(\mathbf{x}) \\
\leq& \begin{cases} C\|\Delta\mathbf{x}\|_\infty 2^{-r} + C_r\|\Delta\mathbf{x}\|^2_\infty, & \ell=1 \\ C\|\Delta^\ell\mathbf{x}\|_\infty 2^{-\ell r} + 2C_r\Omega_\ell(\mathbf{x}), & \ell=2,\dots,s \end{cases} \\
\leq& \begin{cases} C\|\Delta\mathbf{x}\|_\infty + C_r\|\Delta\mathbf{x}\|_\infty, & \ell=1 \\ C\|\Delta^\ell\mathbf{x}\|_\infty + 2C_r\Omega_{\ell-1}(\mathbf{x}), & \ell=2,\dots,s \end{cases} .
\end{aligned}
\tag{3.21}
$$

In the last inequality above, we assume $\|\Delta\mathbf{x}\|_\infty \leq 1$ for both parts. For the case of $\ell=1$, we of course have $\|\Delta\mathbf{x}\|^2_\infty \leq \|\Delta\mathbf{x}\|_\infty$. For the $\ell \geq 2$ case, we use $\|\Delta^t\mathbf{x}\|_\infty \leq 2\|\Delta^{t-1}\mathbf{x}\|_\infty$ and $\|\Delta\mathbf{x}\|_\infty \leq 1$ to deduce $\Omega_\ell(\mathbf{x}) \leq 2\Omega_{\ell-1}(\mathbf{x})$. (3.20) then follows from (3.21) and the definition of $\Omega_\ell(\mathbf{x})$.

The proof of (b) follows verbatim from the one given for part (a) after replacing $\mathbf{x}$, Q, and $Q_{\rm lin}$ by $\mathbf{x}$ (now in ℓ^∞), S, and $S_{\rm lin}$, respectively. □

The following result gives decay estimates on the individual components of $Q^j(\boldsymbol{\delta})$.

Proposition 3.22. *Assume S satisfies the compatibility and the order s differential proximity condition. Assume also that $S_{\rm lin}$ is L_∞-stable and $C^{s,\alpha}$ smooth. Let $\delta^{(j)}_\ell$ denote the iterates*

$$(\delta^{(j)}_0, \delta^{(j)}_1, \dots, \delta^{(j)}_s, \dots, \delta^{(j)}_K) := \Psi^j(\delta^{(0)}_0, \delta^{(0)}_1, \dots, \delta^{(0)}_s, \dots, \delta^{(0)}_K),$$

$j = 0,1,2,\dots$ for initial data with $\delta^{(0)}_\ell$, $\ell \geq 1$, small enough. Then for any $\epsilon > 0$, there is a constant C_ϵ, independent of $\delta^{(0)}_1, \dots, \delta^{(0)}_K$ but may be dependent on $x^{(0)}$, such that

$$
\|\delta^{(j)}_\ell\| \leq C_\epsilon \cdot \begin{cases} 2^{-(\ell-\epsilon)j} \sum_{{\rm weight}(\nu)\geq \ell} \left\|\delta^{(0)}\right\|^\nu, & \ell = 1,\dots,s, \\ 2^{-(s+\alpha-\epsilon)j} \sum_{{\rm weight}(\nu)\geq s+1} \left\|\delta^{(0)}\right\|^\nu, & \ell = s+1,\dots,K. \end{cases}
\tag{3.23}
$$

In the above, we use the shorthand $\|\delta^{(0)}\|^\nu := \|\delta^{(0)}_1\|^{\nu_1} \cdots \|\delta^{(0)}_K\|^{\nu_K}$, and the summations on the right-hand side of (3.23) range over a finite number of multi-indices ν.

Proof. We first show that the assumptions in Proposition 3.22 imply those in Proposition 3.1(a). We then show that the hypotheses of Proposition 3.1(a) imply the conclusion of Proposition 3.22.

We begin with an observation relating the differences δ_ℓ to the components of the sequence $\Delta^\ell\mathbf{x}$. Notice that, by definition (see (1.13)), δ_ℓ is the 0-th entry in the sequence $(\Delta^\ell\mathbf{x})_r$, $r = 0,\dots,K-\ell$. We, therefore, have the trivial bound

$$\|\delta_\ell\| \leq \|\Delta^\ell\mathbf{x}\|_\infty,$$

which in turn implies that

$$\|\delta\|^\nu = \|\delta_1\|^{\nu_1} \cdots \|\delta_K\|^{\nu_K} = O(\Omega_{{\rm weight}(\nu)}(\mathbf{x})) = O(\Omega_\ell(\mathbf{x})). \tag{3.24}$$

Notice also that for $0 \leq r \leq K$,

$$(\Delta^\ell\mathbf{x})_r = \delta_\ell + (\text{a linear combination of } \delta_{\ell'},\ \ell' > \ell)\,, \tag{3.25}$$

for any multi-indexν with ${\rm weight}(\nu) \geq \ell+1$ and $|\nu| \geq 2$.

We may now estimate as follows using the differential proximity condition and (3.24):

$$\|\Psi_\ell(\delta) - \Psi_{\mathrm{lin},\ell}(\delta)\| \lesssim \sum_{\mathrm{weight}(\nu)\geq \ell+1} \|\delta\|^\nu = O(\Omega_\ell(\mathbf{x})). \tag{3.26}$$

But (3.25) implies the equality

$$(\Delta^\ell Q\mathbf{x} - \Delta^\ell Q_{\mathrm{lin}}\mathbf{x})_r = \Psi_\ell(\delta) - \Psi_{\mathrm{lin},\ell}(\delta) + (\text{a linear combination of } \Psi_{\ell'}(\delta) - \Psi_{\mathrm{lin},\ell'}(\delta),\ \ell' > \ell)\,.$$

Consequently, by (3.26)

$$\max_r \|(\Delta^\ell Q\mathbf{x} - \Delta^\ell Q_{\mathrm{lin}}\mathbf{x})_r\| = O(\Omega_\ell(x)) + \sum_{\ell' > \ell} O(\Omega_{\ell'}(x)) = O(\Omega_\ell(x)).$$

We next claim that the assumption that S_{lin} is stable and the $C^{s,\alpha}$-condition imply the estimate (3.2a), as well as the estimate (3.2b), when $s < K$. For, by the stability argument in Section 5, when S_{lin} is stable and $C^{s,\alpha}$, we have $\|\Delta^\ell S^j_{\mathrm{lin}}\mathbf{x}\|_{\ell^\infty} \leq C(\mathbf{x})2^{-j\ell}$, $\ell = 1, \ldots, s$ and $\|\Delta^{s+1} S^j_{\mathrm{lin}}\mathbf{x}\|_{\ell^\infty} \leq C(\mathbf{x})2^{-j(s+\alpha)}$, for some constant $C(\mathbf{x})$ depending on $\mathbf{x}$ but independent of j. Next, we use the well-known fact from the linear theory that there exists a so-called derived subdivision scheme $S^{[\ell]}_{\mathrm{lin}}$ such that $\Delta^\ell \circ S_{\mathrm{lin}} = S^{[\ell]}_{\mathrm{lin}} \circ \Delta^\ell$. We now have

$$\left\|(S^{[\ell]}_{\mathrm{lin}})^j \Delta^\ell \mathbf{x}\right\|_{\ell^\infty} = \left\|\Delta^\ell S^j_{\mathrm{lin}}\mathbf{x}\right\|_{\ell^\infty} \leq \begin{cases} C(\mathbf{x})2^{-j\ell}, & \ell \leq s; \\ C(\mathbf{x})2^{-j(s+\alpha)}, & \ell = s+1. \end{cases}$$

To get the desired conditions, exploit the fact that whence the minimal invariant neighborhood of S_{lin} has a size of $K+1$, that of $S^{[\ell]}_{\mathrm{lin}}$ is $K+1-\ell$. Moreover, if we denote by $Q^{[\ell]}_{\mathrm{lin}}$ the restriction of $S^{[\ell]}_{\mathrm{lin}}$ to such an invariant neighborhood, we have $\Delta^\ell \circ Q_{\mathrm{lin}} = Q^{[\ell]}_{\mathrm{lin}} \circ \Delta^\ell$. When $\mathbf{x}$ is a length $K+1$ sequence, we have, for $\ell \leq s$,

$$\|\Delta^\ell Q^j_{\mathrm{lin}}\mathbf{x}\|_\infty = \|(Q^{[\ell]}_{\mathrm{lin}})^j \Delta^\ell \mathbf{x}\|_\infty \leq C(\mathbf{x})2^{-j\ell}.$$

Since $\mathbf{y} := \Delta^\ell \mathbf{x}$ can be any (length $K+1-\ell$)) sequence, by the uniform boundedness principle, the operator norms of $\{2^{j\ell}(Q^{[\ell]}_{\mathrm{lin}})^j : j \geq 1\}$ are uniformly bounded, i.e. $\|2^{j\ell}(Q^{[\ell]}_{\mathrm{lin}})^j \mathbf{y}\|_\infty \leq C\|\mathbf{y}\|_\infty$ for some constant $C > 0$ independent of $\mathbf{y}$ and of j. This proves (3.2a).

The proof for (3.2b) is similar, provided the support size $K+1$ is large enough to accommodate at least one entry of $\Delta^{s+1}\mathbf{x}$, i.e. when $K > s$.

It remains to see that the conclusion of Proposition 3.1(a) implies the conclusion of Proposition 3.22, i.e. we need to show that (3.3a) implies the first half of (3.23) (pertaining to $\ell \leq s$) and, when $K > s$, (3.3b) implies the second half of (3.23). But these follow again from (3.24) and (3.25). □

4. Proof of Sufficiency

In this section we prove that the compatibility condition and the order k differential proximity condition together imply the strong proximity condition of order k. We begin with a weaker result, which we then bootstrap to obtain the strong proximity condition. The bootstrapping argument relies on a result that we call the *Alternating Sign Lemma*, which reveals a subtle structure enjoyed by nonlinear subdivision rules.

Proposition 4.1. *The smooth compatibility condition implies the strong proximity condition of order* 1*, i.e.* $\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty \leq C\|\Delta\mathbf{x}\|_\infty^2$*. The order* k *differential proximity condition implies the weak proximity condition of order* $k-1$*.*

Proof. The first claim follows from the locality of the subdivision scheme S and a basic Taylor expansion.[6]

To show that the second claim holds, let $\mathbf{x} : \mathbb{Z} \to \mathbb{R}^n$ be a sufficiently dense bounded sequence. We have to estimate the difference $(\Delta^j S\mathbf{x})_l - (\Delta^j S_{\text{lin}}\mathbf{x})_l$, for $j = 1, \ldots, k-1$. Recall that S and S_{lin} map any $K+1$ consecutive entries of $\mathbf{x}$ to exactly $K+1$ entries of $S\mathbf{x}$ and $S_{\text{lin}}\mathbf{x}$, respectively. For an arbitrary index l, we need the $j+1$ entries $(\Delta^j S\mathbf{x})_l, \ldots, (\Delta^j S\mathbf{x})_{l+j}$ to determine $(\Delta^j S\mathbf{x})_l$. By (1.9), there are two cases to consider:

(i) l has the same parity as the shift factor s. Choose i_0 so that $l = 2i_0 + s$. Then by (1.9),

$$((S\mathbf{x})_l, \ldots, (S\mathbf{x})_{l+j}) = (y_{2i_0+s}, \ldots, y_{2i_0+s+j}). \tag{4.2}$$

(ii) l has the opposite parity of s. Choose i_0 so that $l = 2i_0 + s + 1$. Then

$$((S\mathbf{x})_l, \ldots, (S\mathbf{x})_{l+j}) = (y_{2i_0+s+1}, \ldots, y_{2i_0+s+j+1}). \tag{4.3}$$

Since $K \geq k$, and $j \leq k-1$, then either (4.2) or (4.3) is computable from the output of $Q([x_{i_0}, \ldots, x_{i_0+K}])$ A similar comment applies to $(\Delta^j S_{\text{lin}}\mathbf{x})_l$.

Now recall the definition of the map Ψ. Set $\delta_j := \Delta^j(x_{i_0}, \ldots, x_{i_0+j})$. In case (i) above,

$$(\Delta^j S\mathbf{x})_l = \Delta^j(y_l, \ldots, y_{l+j}) = \Psi_j\Big(x_{i_0}, \delta_1, \ldots, \delta_K\Big),$$

while in case (ii),

$$\begin{aligned}(\Delta^j S\mathbf{x})_l =& \Delta^j(y_{l+1}, \ldots, y_{l+j+1}) \\ =& \Delta^j(y_l, \ldots, y_{l+j}) + \Delta^{j+1}(y_l, \ldots, y_{l+j+1}) \\ =& \Psi_j\Big(x_{i_0}, \delta_1, \ldots, \delta_K\Big) + \Psi_{j+1}\Big(x_{i_0}, \delta_1, \ldots, \delta_K\Big).\end{aligned}$$

Similarly, in case (i)

$$(\Delta^j S_{\text{lin}}\mathbf{x})_l = \Psi_{\text{lin},j}\Big(x_{i_0}, \delta_1, \ldots, \delta_K\Big),$$

while in case (ii)

$$(\Delta^j S_{\text{lin}}\mathbf{x})_l = \Psi_{\text{lin},j}\Big(x_{i_0}, \delta_1, \ldots, \delta_K\Big) + \Psi_{\text{lin},j+1}\Big(x_{i_0}, \delta_1, \ldots, \delta_K\Big).$$

[6] This observation motivates the C^1 proximity condition that first appeared in [22].

For $1 \leq j \leq k$, $\Psi_j(x_{i_0}, 0, \dots, 0) = 0$ and we have the Taylor expansion

$$\begin{aligned}
\Psi_j(x_{i_0}, \delta_1, \dots, \delta_K) &= \sum_{|\nu|=1}^{j} \frac{1}{\nu!} D^\nu \Psi_j|_{(x_{i_0},0,\dots,0)}(\delta_1^{\nu_1}, \dots, \delta_K^{\nu_K}) + O\left(\sum_{|\nu|=j+1} \|\delta_1\|^{\nu_1} \cdots \|\delta_K\|^{\nu_K}\right), \\
&= \sum_{|\nu|=1} D^\nu \Psi_j|_{(x_{i_0},0,\dots,0)}(\delta_1^{\nu_1}, \dots, \delta_K^{\nu_K}) + \sum_{\substack{|\nu|=2,\dots,j \\ \mathrm{weight}(\nu)>j}} \frac{1}{\nu!} D^\nu \Psi_j|_{(x_{i_0},0,\dots,0)}(\delta_1^{\nu_1}, \dots, \delta_K^{\nu_K}) \\
&\qquad + O\left(\sum_{|\nu|=j+1} \|\delta_1\|^{\nu_1} \cdots \|\delta_K\|^{\nu_K}\right).
\end{aligned}$$

By the compatibility condition (1.15), the linear part above cancels with $\Psi_{\mathrm{lin},j}(x_{i_0}, \delta_1, \dots, \delta_K)$. Therefore, in case (i),

$$\begin{aligned}
(\Delta^j S\mathbf{x})_l - (\Delta^j S_{\mathrm{lin}}\mathbf{x})_l &= \Psi_j(x_{i_0}, \delta_1, \dots, \delta_K) - \Psi_{\mathrm{lin},j}(x_{i_0}, \delta_1, \dots, \delta_K) \\
&= O\left(\sum_{\substack{|\nu|=2,\dots,j \\ \mathrm{weight}(\nu)>j}} \|\delta_1\|^{\nu_1} \cdots \|\delta_K\|^{\nu_K}\right) + O\left(\sum_{|\nu|=j+1} \|\delta_1\|^{\nu_1} \cdots \|\delta_K\|^{\nu_K}\right) \\
&= O(\Omega_j(\mathbf{x})).
\end{aligned}$$

while in case (ii)

$$\begin{aligned}
(\Delta^j S\mathbf{x})_l - (\Delta^j S_{\mathrm{lin}}\mathbf{x})_l &= \Big[\Psi_j(x_{i_0}, \delta_1, \dots, \delta_K) - \Psi_{\mathrm{lin},j}(x_{i_0}, \delta_1, \dots, \delta_K)\Big] \\
&\qquad + \Big[\Psi_{j+1}\Big(x_{i_0}, \delta_1, \dots, \delta_K\Big) - \Psi_{\mathrm{lin},j+1}\Big(x_{i_0}, \delta_1, \dots, \delta_K\Big)\Big] \\
&= O(\Omega_j(\mathbf{x})) + O(\Omega_{j+1}(\mathbf{x})) \\
&= O(\Omega_j(\mathbf{x})).
\end{aligned}$$

We note that in the second step of equality above, the assumption $j + 1 \leq k$ is essential.

Combining cases (i) and (ii) yields the weak proximity condition (1.31) of order $k - 1$. □

Remark 4.4. Proposition 4.1 does not prove the sufficiency part of our main result Theorem 1.19, for the simple argument above fails to prove the highest desired order (i.e. k) of proximity condition. The proof of Proposition 4.1, does however suggest a way to remedy this problem. If K is at least $k + 1$, then we could impose the additional assumption

$$D^\nu \Psi_{k+1}|_{(x_0,0,\dots,0)} = 0, \quad |\nu| \geq 2, \quad \mathrm{weight}(\nu) \leq k \tag{4.5}$$

to (1.17) in Definition 1.16. With this additional condition, the argument above will allow us to conclude the weak proximity condition of order k.

There is another problem, however: if $K = k$ and $j = k$, then the right-hand side of (4.3) in case (ii) above is dependent on $K + 2$ consecutive entries of $\mathbf{x}$, and therefore cannot be determined from the output of Q regardless of the input. Figure 4 illustrates the problem in the case $k = K = 2$.) We could avoid this problem when $K = k$ by redefining the self-map Q to map $k + 2$ entries of $\mathbf{x}$ to $k + 2$ entries of $S\mathbf{x}$, and redefine Ψ accordingly. For example, for a scheme compatible with the cubic ($k = 2$) B-spline scheme, as shown in Figure 1, define Q_{big}, Ψ_{big} by

$$Q_{\text{big}}(x_0, x_1, x_2, x_3) = \begin{bmatrix} q_1(x_0, x_1) \\ q_0(x_0, x_1, x_2) \\ q_1(x_1, x_2) \\ q_0(x_1, x_2, x_3) \end{bmatrix}, \quad \Psi_{\text{big}} = \Delta \circ Q_{\text{big}} \circ \Sigma.$$

Notice $\Psi_{\text{big},\ell} = \Psi_\ell$ for $1 \le \ell \le k$. We impose (1.17) on components 1 through k of Ψ_{big}, and condition (4.5) on the $(k+1)$-th component of Ψ_{big}. Under this extended (*and apparently stronger*) condition based on Ψ_{big}, the order k weak proximity condition can be concluded using essentially the same argument used to prove Proposition 4.1.

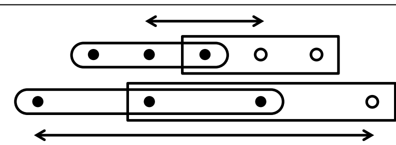

FIGURE 4. When $k = K = 2$, the second order difference of the three data points marked by $\longleftrightarrow$ in the upper row (representing finer level data) depends on 4 ($= K + 2$) data points at the coarser level.

In light of the above remarks, it appears that our differential proximity condition may be too weak to imply the weak proximity condition in general, for it appears that we have to impose a condition on the $(k+1)$-st component of Ψ and, in some cases extend the map Ψ.

In fact, quite the opposite it true: as Theorem 4.13 below shows, the differential proximity condition already implies the *strong* proximity condition.

Notation 4.6. To facilitate the proof of Theorem 4.13 we introduce the following notation. We denote the components of the map $Q \circ \Sigma$ by y_i, i.e.

$$Q \circ \Sigma(x_0, \delta_1, \dots, \delta_K) = (y_0, y_1, y_2, \dots, y_K).$$

We refer to the forward differences formed by the components as

$$(\Delta^\ell y)_i = \Delta^\ell(y_i, y_{i+1}, \dots, y_{i+\ell}).$$

Recall that $\Sigma(x_0, \delta_1, \dots, \delta_K) = (x_0, x_1, \dots, x_K)$ means $\delta_\ell = \Delta^\ell(x_0, x_1, \dots, x_\ell)$. We view each y_i, and hence also each $(\Delta^\ell y)_i$, as a smooth function of the variables $x_0, \delta_1, \dots, \delta_K$. Note that Ψ_ℓ, i.e. the ℓ-th component of the map Ψ, is the same as $(\Delta^\ell y)_0$.

In addition to the compatibility condition, a subtle recurrence structure in the Taylor expansion of $(\Delta^\ell y)_i$ plays a role in the proof of the theorem. The next lemma makes this structure explicit. The proof of the lemma relies on *both* the subdivision structure and the smooth compatibility condition satisfied by the map Q.

Lemma 4.7 (**Alternating sign lemma**). *Let $\delta^\nu := \delta_1^{\nu_1} \cdots \delta_K^{\nu_K}$ and, for $|\nu| > 1$, let $c^\ell_{i,\nu}(x_0)$ be the coefficient of δ^ν in the Taylor expansion of $(\Delta^\ell y)_i$ in the variables $\delta_1, \dots, \delta_K$, i.e. we have the formal power series:*

$$(\Delta^\ell y)_i = \sum_{|\nu| \ge 0} c^\ell_{i,\nu}(x_0)\delta^\nu,$$

then

(1) $c^1_{i,\nu}(x_0) + c^1_{i+1,\nu}(x_0) = 0$ *for all* i, x_0 *and* $\mathrm{weight}(\nu) = 2$ *(i.e.* $\nu = (2, 0, \ldots)$*);*
(2) *for any* $\ell \geq 2$ *and* ν *with* $\mathrm{weight}(\nu) > 2$, *if* $c^{\ell-1}_{i,\xi}(x_0) = 0$ *for all* i, x_0 *and* ξ *with* $\mathrm{weight}(\xi) < \mathrm{weight}(\nu)$, *then*

$$c^\ell_{i,\nu}(x_0) + c^\ell_{i+1,\nu}(x_0) = 0$$

for any i, x_0.
(2′) *for any* $\ell \geq 3$, *if* $c^{\ell-2}_{i,\nu}(x_0) = 0$ *for all* i, x_0 *and* ν *with* $\mathrm{weight}(\nu) \leq \ell - 1$, *then*

$$c^{\ell-1}_{i,\nu}(x_0) + c^{\ell-1}_{i+1,\nu}(x_0) = 0$$

for any i, x_0 *and any* ν *with* $\mathrm{weight}(\nu) \leq \ell$. *(Note: (2′) is special case of (2).)*

Proof. Since $(\Delta^\ell y)_i = (\Delta^{\ell-1} y)_{i+1} - (\Delta^{\ell-1} y)_i$, we have

$$(\Delta^\ell y)_i + (\Delta^\ell y)_{i+1} = (\Delta^{\ell-1} y)_{i+2} - (\Delta^{\ell-1} y)_i. \tag{4.8}$$

Our goal is to show that $c^\ell_{i,\nu}(x_0) + c^\ell_{i+1,\nu}(x_0) = 0$ when $\ell = \mathrm{weight}(\nu) - 1$ and under the (inductive) assumptions in the lemma statement. Note that $c^\ell_{i,\nu}(x_0) + c^\ell_{i+1,\nu}(x_0)$ is the coefficient of δ^ν in the Taylor expansion of (4.8) at $(x_0, 0, \ldots)$, viewing (4.8) as a function of x_0, $\delta_1, \delta_2, \ldots$.

By definition, the Taylor expansion of $(\Delta^{\ell-1} y)_i$ about $(x_0, 0, \ldots)$ is

$$(\Delta^{\ell-1} y)_i = \text{linear terms} + \sum_{|\nu| \geq 2} c^{\ell-1}_{i,\nu}(x_0)\delta^\nu. \tag{4.9}$$

Notice also that the compatibility condition implies the first partial derivatives $c^0_{i,s} = \dfrac{\partial y_i(x_0, 0, \ldots, 0)}{\partial \delta_s}$, $1 \leq s \leq K$ are independent of x_0. Thus, we have the following Taylor expansion

$$y_i = x_0 + \sum_{s=1}^K c^0_{i,s}\delta_s + \sum_{|\nu| \geq 2} c^0_{i,\nu}(x_0)\delta_1^{\nu_1}\delta_2^{\nu_2} \cdots$$

where $c^0_{i,s}$ are constants independent of x_0. It follows that the linear terms in (4.9) are independent of x_0.

We now exploit the underlying subdivision structure: if the input sequence $\mathbf{x}$ is shifted by one entry, then the subdivided sequence $S\mathbf{x}$ is shifted by **two** entries (see Equation (1.4)). Since $x_1 = x_0 + \delta_1$,

$$y_{i+2}(x_0, \delta_1, \delta_2, \ldots) = y_i(x_1, \delta_1 + \delta_2, \ldots) = y_i(x_0 + \delta_1, \delta_1 + \delta_2, \delta_2 + \delta_3, \ldots),$$

for all x_0. We may now compute the Taylor expansion of $(\Delta^{\ell-1}y)_{i+2}$ for $\ell > 0$ by first expanding about $(x_1, 0, \ldots, 0)$ and then setting $x_1 = x_0 + \delta_1$ and expanding in δ about $(x_0, 0, \ldots, 0)$:

(4.10)
$$\begin{aligned}(\Delta^{\ell-1}y)_{i+2} &= \text{linear terms} + \sum_{|\nu|\geq 2} c^{\ell-1}_{i,\nu}(x_1)(\delta_1+\delta_2)^{\nu_1}(\delta_2+\delta_3)^{\nu_2}\cdots \\ &= \text{linear terms} + \sum_{|\nu|\geq 2}\left\{c^{\ell-1}_{i,\nu}(x_0) + \sum_{|\nu_0|>0} c^{\ell-1}_{i,\nu,\nu_0}(x_0)\,\delta_1^{\nu_0}\right\}(\delta_1+\delta_2)^{\nu_1}(\delta_2+\delta_3)^{\nu_2}\cdots \\ &= \text{linear terms} + \sum_{|\nu|\geq 2}\left[c^{\ell-1}_{i,\nu}(x_0)\delta^{\nu} + \sum_{|\nu_0|\geq 0}\sum_{\theta\in\Theta(\nu_0,\nu)} c^{\ell-1}_{i,\nu,\nu_0}(x_0)\,\delta^{\theta}\right] \\ &= \text{linear terms} + \sum_{|\nu|\geq 2}\left[c^{\ell-1}_{i,\nu}(x_0)\delta^{\nu} + \left\{\sum_{(\nu_0,\xi)\in\Xi(\nu)} c^{\ell-1}_{i,\xi,\nu_0}(x_0)\right\}\delta^{\nu}\right].\end{aligned}$$

Notice that because the linear terms in the first line above are independent of x_1, they contribute no nonlinear terms to the second line above. In the penultimate line, each $\Theta(n,\nu)$ is a finite set of multi-indices with weights *strictly greater than* weight(ν). The last line is obtained by rearranging terms so that, for each fixed ν, $|\nu| \geq 2$, a finite set $\Xi(\nu)$ of multi-indices (ν_0, ξ), with $|\nu_0| \geq 0$ and $2 \leq \text{weight}(\xi) < \text{weight}(\nu)$ so that the last line holds. Now, by (4.8)-(4.10),

$$\begin{aligned}(\Delta^{\ell}y)_i + (\Delta^{\ell}y)_{i+1} &= (\Delta^{\ell-1}y)_{i+2} - (\Delta^{\ell-1}y)_i \\ &= \text{linear terms } + \sum_{|\nu|\geq 2}\sum_{(n,\xi)\in\Xi(\nu)} (c^{\ell-1}_{i,\xi}(x_0))^{(n)}\delta^{\nu},\end{aligned} \tag{4.11}$$

or

$$c^{\ell}_{i,\nu}(x_0) + c^{\ell}_{i+1,\nu}(x_0) = \sum_{(\nu_0,\xi)\in\Xi(\nu)} c^{\ell-1}_{i,\xi,\nu_0}(x_0)(x_0). \tag{4.12}$$

When weight$(\nu) = 2$, $\Xi(\nu)$ is empty and part (1) of the lemma follows. Part (2) follows from the relation (4.12). Part (2′) is just a special case of (2). ☐

Theorem 4.13. *The order k differential proximity condition implies the strong order k proximity condition.*

Proof. For $k = 1$, Theorem 4.13 follows immediately from the compatibility assumption.

Suppose $k \geq 2$. In the notation above, the differential proximity condition gives:

$$\begin{gathered}\frac{1}{\nu!}D^{\nu}\Psi_{\ell}|_{(x_0,0,\ldots,0)} = c^{\ell}_{0,\nu}(x_0) = c^{\ell-1}_{1,\nu}(x_0) - c^{\ell-1}_{0,\nu}(x_0) = 0, \\ \text{weight}(\nu) \leq \ell,\ \ \ell = 2, \ldots, k.\end{gathered} \tag{4.14}$$

By virtue of the argument in Proposition 4.1, the strong order k ($k \geq 2$) proximity condition follows if we can show:

$$c^{\ell-1}_{0,\nu}(x_0) = 0 = c^{\ell-1}_{1,\nu}(x_0), \quad \text{weight}(\nu) \leq \ell,\ \ \ell = 2, \ldots, k. \tag{4.15}$$

This follows from the following stronger statement:

$$c^{\ell-1}_{i,\nu}(x_0) = 0,\ \ \forall\, i,\ \text{weight}(\nu) \leq \ell,\ \ \ell = 2, \ldots, k\,, \tag{4.16}$$

which we prove by induction on ℓ:

If $\ell = 2$, then (4.15) follows from (4.14) and part (1) of the Alternating Sign Lemma. But then (4.15) combined with part (1) imply (4.16).

If $\ell = 3$ (when $k \geq 3$), we need to prove $c^2_{i,\nu}(x_0) = 0$ for all ν with weight 2 or 3. This follows from part (2′) of the Alternating Sign Lemma, as the assumption in part (2′) was established in the previous $\ell = 2$ step.

Applying the Alternating Sign Lemma completes the induction step. □

As a corollary of the proof, we have:

Corollary 4.17. *The differential proximity condition in Definition 1.16 is equivalent to the apparently stronger condition* (1.18)*.*

Proof. If $K > k$,

$$D^\nu \Psi_{k+1}|_{(x_0,0,\ldots,0)} = c^{k+1}_{0,\nu}(x_0) = c^{k-1}_{2,\nu}(x_0) - 2c^{k-1}_{1,\nu}(x_0) + c^{k-1}_{0,\nu}(x_0) \overset{(4.16)}{=} 0, \;\; \forall \text{ weight}(\nu) \leq k.$$

Similarly,

$$D^\nu \Psi_{k+k'}|_{(x_0,0,\ldots,0)} = c^{k+k'}_{0,\nu}(x_0) = \sum_{j=0}^{k'+1} (-1)^{k'+1-j} \binom{k'+1}{j} c^{k-1}_{j,\nu}(x_0) \overset{(4.16)}{=} 0, \;\; \forall \text{ weight}(\nu) \leq k.$$

□

At this point, we have shown that the compatibility condition together with the order k differential proximity condition implies the strong proximity condition of order k. Recall that [28, Theorem 2.4] states that the strong proximity condition of order k implies that the subdivision rule S is C^k, this completes the proof of sufficiency.

5. Proof of Necessity

To prove the necessity part of Theorem 1.19, we need to argue that when the order k differential proximity condition is not satisfied by S, then S cannot be C^k smooth.

The necessity proof is based on two key results Theorems 5.13 and 5.48, whose proofs we defer to Sections 5.1 and 5.2, respectively.

Assume that S is compatible with an L_∞-stable, C^k linear subdivision rule S_{lin}. Without loss of generality, assume that S satisfies the order $k-1$ but not the order k differential proximity condition with S_{lin}. To prove necessity, we argue as follows:

(I) Theorem 5.13 states that when S does not satisfy the order k proximity condition with S_{lin}, then the inequality

$$\|\Delta^k S^j c\|_\infty \gtrsim j 2^{-kj} \tag{5.1}$$

is satisfied for at some sufficiently dense initial control data c. Call the corresponding limit function F.

(II) By part (b) of Theorem 5.48 there is a constant τ such that the following estimate is satisfied:

$$\sup_{i\in\mathbb{Z}} \left| 2^{j(k-1)}(\Delta^{k-1}S^j c)_i - F^{(k-1)}(2^{-j}(i+\tau)) \right|_\infty \lesssim 2^{-j}. \tag{5.2}$$

We claim that (I) and (II) imply that $F^{(k-1)}$ cannot be Lipschitz. For assume the contrary, then by the triangle inequality we may estimate as follows:

$$\begin{aligned}
\left\| 2^{kj}(\Delta^k S^j c)_i \right\| &= \quad 2^j \left\| 2^{(k-1)j}(\Delta^{k-1}S^j c)_{i+1} - 2^{(k-1)j}(\Delta^{k-1}S^j c)_i \right\| \\
&\le \ 2^j \underbrace{\left\| (\Delta^{k-1}S^j c)_{i+1} - F^{(k-1)}(2^{-j}(i+1+\tau)) \right\|}_{=O(2^{-j}) \text{ by [II]}} \\
&\quad + 2^j \underbrace{\left\| F^{(k-1)}(2^{-j}(i+1+\tau)) - F^{(k-1)}(2^{-j}(i+\tau)) \right\|}_{=O(2^{-j}) \text{ by the Lipschitz assumption}} \\
&\quad + 2^j \underbrace{\left\| F^{(k-1)}(2^{-j}(i+\tau)) - (\Delta^{k-1}S^j c)_i \right\|}_{=O(2^{-j}) \text{ by [II]}} \\
&= \quad O(1).
\end{aligned}$$

This contradicts (5.1) in [I]. Hence, F cannot be $C^{k-1,1}$ smooth, let alone C^k.

To conclude the proof of necessity, it only remains to justify (I) and (II).

5.1. **Proof of (I): Dynamical System Resonance.** Condition (5.1) in [I] is a result pertaining to the *decay rate* of differences in subdivision data. As we alluded to in the introduction, when the order k proximity condition is violated, resonance effects occur that slow the decay of the right-hand side of (5.1) from $O(2^{-kj})$ (had the proximity condition been satisfied) to $O(j2^{-kj})$. Such resonance phenomena are well known in the literature on dynamical systems and are known to be caused by the presence of so-called *resonance terms*. However, proving the required *lower bound* is technical. It requires a delicate argument to show that one can choose initial data so that the effect of resonance terms would not dissipate in the course of iteration.

To prove (I), we must study the decay properties of the different components of $y^{(j)} := \Psi^j\left(y^{(0)}\right)$ under the assumption that the underlying linear scheme S_{lin} is L_∞-stable and C^k. From the theory of linear subdivision schemes, the (common) spectrum of Q_{lin} and Ψ_{lin} – see (1.15) – has leading eigenvalues $\lambda_\ell := 1/2^\ell$, $\ell = 0, 1, \ldots, k$. In dynamical system jargon, this set of dyadic eigenvalues is "resonance prone", i.e. for any $\ell \ge 2$ there always exist $\nu = (\nu_1, \ldots, \nu_k)$ with $|\nu| \ge 2$ such that

$$\lambda_\ell = \lambda_1^{\nu_1} \cdots \lambda_k^{\nu_k}, \text{ or, equivalently, } \ell = \sum_i i\nu_i = \text{weight}(\nu).$$

There is an abundance of such ν's when ℓ is large: in fact the set $\Gamma_{\ell-1}$ defined in (1.27) enumerates *all* the possibilities. Therefore, *violation of the order k proximity condition corresponds exactly to* *the presence of resonance in the k-th component.* Theorem 5.13 show that for at least some choice of initial data, the decay rate of the k-th component must be slower than what it would have been if the order k proximity condition were satisfied.

Our proof relies on three technical lemmas.

Lemma 5.3. *Let $v_j = \lambda v_{j-1} + r_{j-1} + y_{j-1}$ where $|y_j| \le C_1\, \mu^j$, $0<\mu<\lambda<1$, $C_1>0$, and*

$$r_j \ge C_0\, \lambda^j \quad (\text{or } r_j \le -C_0\, \lambda^j), \;\; C_0>0.$$

Then

$$|v_j| > C\, j\, \lambda^j, \quad \forall\, j \ge j_0,$$

for some constant $C>0$ and some large enough j_0.

Proof. By iterating the recurrence $v_j = \lambda v_{j-1} + r_{j-1} + y_{j-1}$, we have

$$v_j = v_0\lambda^j + \sum_{i=0}^{j-1} \lambda^{j-1-i}(r_i+y_i) = \lambda^{j-1}\sum_{i=0}^{j-1}\lambda^{-i} r_i + \lambda^{j-1}\sum_{i=0}^{j-1}\lambda^{-i}y_i + v_0\lambda^j.$$

Since $|y_j|\le C_1\mu^j$ and $r_j \ge C_0\lambda^j$ or $r_j \le -C_0\lambda^j$, it follows that

$$\begin{aligned}
|v_j| &\ge \underbrace{\left|\lambda^{j-1}\sum_{i=0}^{j-1}\lambda^{-i}r_i\right|}_{=\lambda^{j-1}\sum_{i=0}^{j-1}\lambda^{-i}|r_i|} - \underbrace{\left|\lambda^{j-1}\sum_{i=0}^{j-1}\lambda^{-i}y_i\right|}_{\le\lambda^{j-1}\sum_{i=0}^{j-1}\lambda^{-i}|y_i|} - |v_0\lambda^j| \\
&\ge \lambda^{j-1}\sum_{i=0}^{j-1}\lambda^{-i}(C_0\lambda^i - C_1\mu^i) - |v_0|\lambda^j = C_0 j\lambda^{j-1} - C_1\lambda^{j-1}\sum_{i=0}^{j-1}(\mu\lambda^{-1})^i - |v_0|\lambda^j \\
&\ge C_0 j \lambda^{j-1} - C_1\lambda^{j-1}\sum_{i=0}^{\infty}(\mu\lambda^{-1})^i - |v_0|\lambda^j = \left(C_0\lambda^{-1} - \frac{C_1(\lambda-\mu)^{-1}+|v_0|}{j}\right) j\lambda^j.
\end{aligned}$$

Let $j_0\in\mathbb{N}$ be large enough such that $C := C_0\lambda^{-1} - (C_1(\lambda-\mu)^{-1}+|v_0|)/j_0 > 0$, then $|v_j|\ge Cj\lambda^j$ for $j\ge j_0$. □

Lemma 5.4. *Suppose $v_j = \lambda v_{j-1} + y_{j-1}$. If $|y_j|\le C\mu^j$, $0<\mu<\lambda$ and*

$$C/|v_0| < \lambda-\mu, \tag{5.5}$$

then

$$\Big(|v_0| - \frac{C}{\lambda-\mu}\Big)\lambda^j \le |v_j| \le \Big(|v_0| + \frac{C}{\lambda-\mu}\Big)\lambda^j. \tag{5.6}$$

(The upper bound in (5.6) holds without (5.5).)

Proof. By iterating the recurrence $v_j = \lambda v_{j-1} + y_{j-1}$, we have

$$v_j = \lambda^j v_0 + \sum_{i=0}^{j-1}\lambda^{j-1-i}y_i = \Big(v_0 + \frac{1}{\lambda}\sum_{i=0}^{j-1}\lambda^{-i}y_i\Big)\lambda^j.$$

Since $|y_i| \le C\mu^i$ and $0 < \mu < \lambda$, it follows that

$$|v_j| \le \Big(|v_0| + \frac{1}{\lambda}\sum_{i=0}^{j-1}\lambda^{-i}|y_i|\Big)\lambda^j \le \Big(|v_0| + \frac{C}{\lambda}\sum_{i=0}^{\infty}(\mu/\lambda)^i\Big)\lambda^j = \Big(|v_0| + \frac{C}{\lambda-\mu}\Big)\lambda^j.$$

Similarly, we can lower bound $|v_j|$ as

$$|v_j| \ge \Big(|v_0| - \frac{1}{\lambda}\sum_{i=0}^{j-1}\lambda^{-i}|y_i|\Big)\lambda^j \ge \Big(|v_0| - \frac{C}{\lambda}\sum_{i=0}^{\infty}(\mu/\lambda)^i\Big)\lambda^j = \Big(|v_0| - \frac{C}{\lambda-\mu}\Big)\lambda^j,$$

provided that (5.5) holds. □

Lemma 5.7 (**Stability under iterations**)**.** *Let $U \subset \mathbb{R}^n$ be a neighborhood of $0 \in \mathbb{R}^n$ and let $\Phi : \mathbb{R}^m \times U \to \mathbb{R}^m \times \mathbb{R}^n$ be a C^2 map such that $\Phi(x,0) = (x,0)$ and $D\Phi|_{(x,0)} = \left(\begin{smallmatrix} I & B \\ 0 & A\end{smallmatrix}\right)$ for all x, where A and B are constant matrices, such that all eigenvalues of A have modulus less than unity. Then for any $x^* \in \mathbb{R}^m$ and any $\varepsilon_x, \varepsilon_y > 0$, there exist δ_x, $\delta_y > 0$ such that for all $(x,y) \in \mathbb{R}^n \times U$ with $\|x - x^*\| < \delta_x$ and $\|y\| < \delta_y$ the following conditions are satisfied:*

(i) $\|x^{(j)} - x^*\| < \varepsilon_x$ *and* $\|y^{(j)}\| < \varepsilon_y$ *for all the iterates* $(x^{(j)}, y^{(j)}) = \Psi^j(x,y)$.
(ii) $\lim_{j\to\infty} x^{(j)} = x^{(\infty)}$ *is well-defined and* $\lim_{j\to\infty} y^{(j)} = 0$.
(iii) *The map* $(x,y) \mapsto x^{(\infty)}$ *is Lipschitz.*

Remark 5.8. Notice that we allow the matrix A to be singular. In the special case where A is non-singular, Φ restricts to a diffeomorphism in a neighborhood of its fixed point set and the Stability Lemma is a standard result (the Invariant Manifold Theorem) in dynamical systems theory (see [15]). In fact, the map $(x,y) \mapsto x^\infty$ is actually C^1. When Φ is not a diffeomorphism, the map $(x,y) \mapsto x^\infty$ need not be differentiable. We were unable to find a proof in the literature of the stability lemma when A is allowed to be singular.

Proof. Step 1. It suffices to assume that B is zero. If not, if we can block-diagonalize $D\Phi|_{(x,0)}$ by finding a matrix X so that

$$\begin{aligned}
&\begin{bmatrix} I & B \\ 0 & A\end{bmatrix}\begin{bmatrix} I & X \\ 0 & I\end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & A\end{bmatrix} \\
\iff\ & X - XA = -B \\
\iff\ & (I_{mn} - A^T \otimes I_m)\mathrm{vec}(X) = -\mathrm{vec}(B).
\end{aligned}$$

This linear system of equation has a unique solution because I and A do not have any eigenvalue in common [16, Section 4.4]. It is then easy to verify that if $\tilde{\Phi} := \left(\begin{smallmatrix} I & -X \\ 0 & I\end{smallmatrix}\right) \circ \Phi \circ \left(\begin{smallmatrix} I & X \\ 0 & I\end{smallmatrix}\right)$ satisfies the conclusion of the lemma, then so does Φ. So from now on, we assume $D\Phi|_{(x,0)} = \left(\begin{smallmatrix} I & 0 \\ 0 & A\end{smallmatrix}\right)$.

Step 2. Set $\Phi^q = \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{q-\text{times}}$. Then the hypotheses of the lemma are satisfied by Φ^q for all $q > 0$. Assume for the moment that the conclusions of the lemma hold for the map Φ^q. We claim that they hold for Φ, itself.

To see that (i) is satisfied, choose any ϵ_x and ϵ_y. Because $(x^*, 0)$ is a fixed point of Φ and Φ is continuous, there exist ϵ'_x and ϵ'_y, such that if $\|x - x^*\| < \epsilon'_x$ and $\|y\| < \epsilon'_y$ then $\|x^{(i)} - x^*\| < \epsilon_x$

and $\|y^{(i)}\| < \epsilon_y$ for $(x^{(i)}, y^{(i)}) = \Phi^i(x, y)$, for all $i \le q$. Choose δ_x and δ_y so that $\|x - x^*\| < \delta_x$, $\|y\| < \delta_y$ imply $\|x^{(kq)} - x^*\| < \epsilon'_x$ and $\|y^{(kq)}\| < \epsilon'_y$ for all k.

Suppose that $\|x - x^*\| < \delta_x$ and $\|y\| < \delta_y$, and let j be any positive integer. Then $j = k \cdot q + i$ for some i and k. Then $\|x^{(kq)} - x^*\| < \epsilon'_x$ and $\|y^{(kq)}\| < \epsilon'_y$. Since $(x^{(j)}, y^{(j)}) = \Phi^i(x^{(kq)}, y^{(kq)})$, it follows that $\|x^{(j)} - x^*\| < \epsilon_x$ and $\|y^{(j)}\| < \epsilon_y$.

To see that (ii) and (iii) are satisfied by Φ, it suffices to show

$$\lim_{k\to\infty} \Phi^{kq+i}(x, y) = \lim_{k\to\infty} \Phi^{kq}(x, y)$$

for $0 < i \le q$. Let $(x^*, 0) = \lim_{k\to\infty} \Phi^{kq}(x, y)$. Choose any $\epsilon > 0$. Then since $(x^*, 0)$ is a fixed point of Φ, there is a $\delta > 0$, so that $\|(x, y) - (x^*, 0)\| < \delta$ implies $\|\Phi^i(x, y) - (x^*, 0)\| < \epsilon$ for $i = 1, 2, \ldots, q$. By hypothesis, $\|\Phi^{kq}(x, y) - (x^*, 0)\| < \delta$ for all k sufficiently large. Consequently, $\|\Phi^{kq+i}(x, y) - (x^*, 0)\| = \|\Phi^i(\Phi^{kq}(x, y)) - (x^*, 0)\| < \epsilon$ for all k sufficiently large and all $0 < i \le q$. The result follows.

Step 3. It remains to prove the lemma with Φ replaced by Φ^q for sufficiently large q. Because all eigenvalues of A have modulus less than 1, for q sufficiently large, $\|A^q y\| \le \frac{1}{4}\|y\|$ for all $y \in \mathbb{R}^m$. Since, $D\Phi^q|_{(x^*,0)} = \begin{pmatrix} I & 0 \\ 0 & A^q \end{pmatrix}$, without loss of generality we may assume that $\|Ay\| < \frac{1}{4}\|y\|$ for all y.

For convenience, set $x^* = 0$, let $U_{r_1,r_2} = \{(x, y) : \|x\| < r_1, \|y\| < r_2\}$, and write Φ in the form $\Phi(x, y) = (X(x, y), Y(x, y))$. Since Φ is C^2, the assumptions (i) and (ii), together with Taylor's Theorem, imply that

$$X(x, y) = x + R(x, y) \quad Y(x, y) = Ay + S(x, y)$$

where $R(x, y) = \sum_{i=1}^n \sum_{j=1}^n R_{i,j}(x, y) y^i y^j$ and $S(x, y) = \sum_{i=1}^n \sum_{j=1}^n S_{i,j}(x, y) y^i y^j$ for continuous functions $R_{i,j}$, $S_{i,j}$.

Notice that $R(x, y)$ and $S(x, y)$ are quadratic in y and $R_{i,j}(x, y)$ and $S_{i,j}(x, y)$ are continuous, hence uniformly continuous on the compact set $\overline{U}_{r_1,r_2}$. It follows that there is a constant $C > 0$ such that

$$\|R(\overline{x}, \overline{y}) - R(x, y)\| \le C(\|y\| + \|\overline{y}\|)\|\overline{y} - y\| \text{ and } \|S(\overline{x}, \overline{y}) - S(x, y)\| \le C(\|y\| + \|\overline{y}\|)\|\overline{y} - y\|$$

for all $(x, y), (\overline{x}, \overline{y}) \in \overline{U}r_1, r_2$. For $\overline{y} = 0$ these inequalities reduce to

$$\|R(x, y)\| \le C\|y\|^2 \text{ and } \|S(x, y)\| \le C\|y\|^2$$

Choose $r_2 \le \frac{1}{4C}$ and $r_3 \le \min\left(r_1/2, r_2, \frac{1}{8C}\right)$, and set $U = U_{r_1,r_2}$ and $W = U_{r_1/2,r_3}$.

With these choices, X and Y satisfy the estimates

$$\|X(x, y) - x\| = \|R(x, y)\| \le \frac{1}{4}\|y\| \text{ and } \|Y(x, y)\| \le \|Ay\| + \|S(x, y)\| \le \frac{1}{4}\|y\| + \frac{1}{4}\|y\| = \frac{1}{2}\|y\| \tag{5.9}$$

for all $(x, y) \in \overline{U}$; as well as the estimates

$$\|X(x, y) - X(\overline{x}, \overline{y})\| \le \|x - \overline{x}\| + C2r_3\|y - \overline{y}\| \le \|x - \overline{x}\| + \frac{1}{4}\|y - \overline{y}\| \tag{5.10}$$

and

$$\|Y(x, y) - Y(\overline{x}, \overline{y})\| \le \frac{1}{4}\|y - \overline{y}\| + C2r_3\|y - \overline{y}\| \le \frac{1}{2}\|y - \overline{y}\|\,. \tag{5.11}$$

for all $(x, y), (\overline{x}, \overline{y}) \in \overline{W}$,

Choose $(x, y) \in W$, and set $(x^{(j)}, y^{(j)}) = \Phi^j(x, y)$. Then applying (5.9) inductively starting with $(x^{(0)}, y^{(0)}) = (x, y)$ yields the inequalities

$$\|x^{(j+1)} - x^{(j)}\| \leq \frac{1}{4}\|y^{(j)}\| \text{ and } \|y^{(j+1)}\| \leq \frac{1}{2}\|y^{(j)}\|\,. \tag{5.12}$$

Hence, $\|y^{(j)}\| \leq \frac{1}{2^j}\|y\|$, which in turn implies

$$\|x^{(j)}\| \leq \sum_{k=0}^{j-1} \|x^{(k+1)} - x^{(k)}\| \leq \frac{1}{4}\sum_{k=0}^{j-1} \frac{1}{2^k}\|y\| \leq \frac{1}{2}\|y\|\,.$$

In particular $\|x^{(j)}\| \leq \|x\| + \|x^{(j)} - x\| \leq \frac{1}{2}r_1 + \frac{1}{2}\|y\| \leq \frac{1}{2}r_1$. Part (i) is an immediate consequence, for choose any $\epsilon_x > 0$ and $\epsilon_y > 0$. Then choose $\delta_x = \min(r_1, 2\epsilon_x)$ and $\delta_2 = \min(r_2, \epsilon_2)$.

To prove (ii), observe that the estimates (5.12) imply that $\{x^{(j)}\}$ is a Cauchy sequence, for all $(x, y) \in W$. Consequently, the map

$$\Phi^\infty : W \to \{x \in \mathbb{R}^m \,:\, \|x\| < r_1\} \,:\, (x, y) \mapsto x^{(\infty)} := \lim_{j\to\infty} x^{(j)}$$

is well defined.

That Φ^∞ is Lipschitz follows from (5.10) and (5.11), for choose $(\overline{x}, \overline{y})$ and (x, y) in W, and set $(\overline{x}^{(j)}, \overline{y}^{(j)}) = \Phi^j(\overline{x}, \overline{y})$. Then $\|\overline{y}^{(j)} - y^{(j)}\| \leq \frac{1}{2^j}\|\overline{y} - y\|$ and

$$\|\overline{x}^{(j+1)} - x^{(j+1)}\| \leq \|\overline{x}^{(j)} - x^{(j)}\| + \frac{1}{2^j}\|\overline{y} - y\|\,.$$

Consequently, for all j

$$\|\overline{x}^{(j)} - x\| \leq \|\overline{x} - x\| + \sum_{i=1}^{j-1}\frac{1}{2^i}\|\overline{y} - y\| \leq \|\overline{x} - x\| + \|\overline{y} - y\|\,.$$

Therefore, $\|\overline{x}^{(\infty)} - x^{(\infty)}\| \leq 2\|(\overline{x}, \overline{y}) - (x, y)\|$. □

Theorem 5.13. *Assume that S satisfies the compatibility condition with S_{lin} an L_∞- stable C^k smooth linear subdivision scheme S_{lin}. Suppose further that it satisfies the order $k-1$, but not the order k, differential proximity condition. Then for suitably chosen initial data $[x^{(0)}, \delta_1^{(0)}, \ldots, \delta_k^{(0)}, \ldots, \delta_K^{(0)}]$, the iterates*

$$[x^{(j)}, \delta_1^{(j)}, \ldots, \delta_k^{(j)}, \ldots, \delta_K^{(j)}] := \Psi^j[x^{(0)}, \delta_1^{(0)}, \ldots, \delta_k^{(0)}, \ldots, \delta_K^{(0)}], \tag{5.14}$$

$j = 0, 1, 2, \ldots,$ satisfy

$$\|\delta_k^{(j)}\| \gtrsim j2^{-kj}. \tag{5.15}$$

Proof. It is convenient to divide the proof into four steps.

Step 1. Consider the polynomial

$$\text{Reson}_{\bar{x}}(\delta) = \sum_{\text{weight}(\nu)=k} \frac{1}{\nu!} D^\nu \Psi_\ell|_{(\bar{x},0,\ldots,0)}\delta^\nu.$$

Note that $\text{Reson}_{\bar{x}}$ is a polynomial in the variables $\delta_1, \ldots, \delta_{k-1}$ only. So to avoid confusion, we either write $\text{Reson}_{\bar{x}}(\delta_1, \ldots, \delta_{k-1})$ or change the symbol of independent variable to ω, so whenever we write $\text{Reson}_x(\omega)$, it is understood that $\omega = (\omega_1, \ldots, \omega_{k-1})$ is a vector in $\mathbb{R}^{(k-1)\times n}$.

Our assumption that S does not satisfy the order k differential proximity condition is equivalent to assuming that the polynomial $\text{Reson}_{\bar{x}}$ is non-zero for some $\bar{x}$. Assume $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_{k-1})$ is such that $\text{Reson}_{\bar{x}}(\bar{\omega}) \neq 0$. By continuity, there exist a bounded neighborhood $U_{\bar{x}}$ of $\bar{x}$ (in $\mathbb{R}^n$) and a bounded neighborhood $V_{\bar{\omega}}$ of $\bar{\omega}$ (in $\mathbb{R}^{(k-1)\times n}$) such that

$$\text{Reson}_x(\omega)_i \geq c \text{ or } \leq -c, \quad \forall\, (x, \omega) \in U_{\bar{x}} \times V_{\bar{\omega}} \tag{5.16}$$

for some positive number c and some component $i \in \{1, \ldots, n = \dim(M)\}$.

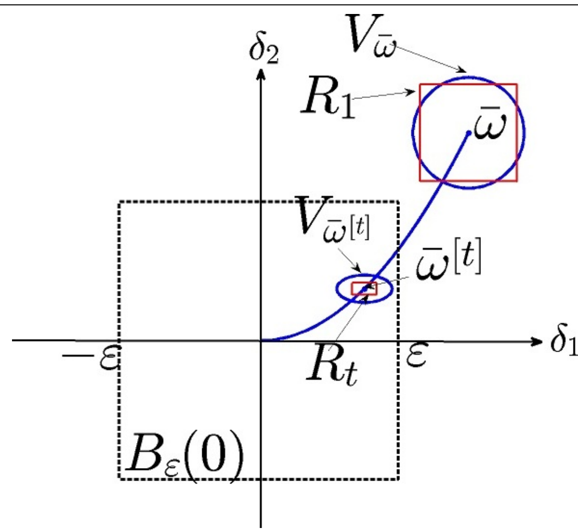

FIGURE 5. Illustration of the various components of the proof of Theorem 5.13

For $\omega = (\omega_1, \ldots, \omega_{k-1})$, let

$$\omega^{[t]} := (t\omega_1, t^2\omega_2, \ldots, t^{k-1}\omega_{k-1}). \tag{5.17}$$

By 'weight-homogeneity',

$$\text{Reson}_x(\omega^{[t]}) = t^k \text{Reson}_x(\omega). \tag{5.18}$$

We also write

$$V_{\bar{\omega}^{[t]}} := \{\omega^{[t]} : \omega \in V_{\bar{\omega}}\}. \tag{5.19}$$

Note that $V_{\bar{\omega}^{[t]}}$ is an open neighborhood of $\bar{\omega}^{[t]}$ in $\mathbb{R}^{(k-1)\times n}$ and

$$\text{Reson}_x(\omega)_i \geq c\, t^k \text{ or } \leq -c\, t^k, \quad \forall\, (x, \omega) \in U_{\bar{x}} \times V_{\bar{\omega}^{[t]}}. \tag{5.20}$$

We now consider the iterates (5.14). Fix $x^{(0)}$ to be the $\bar{x}$ above. By Lemma 5.7, there is an $\varepsilon > 0$ such that if $\|\delta_\ell^{(0)}\| < \varepsilon$, $\ell = 1, \ldots, K$, then

$$x^{(j)} \in U_{\bar{x}}, \ \forall\, j, \text{ and } \|\delta_\ell^{(j)}\| \text{ stays uniformly bounded.} \tag{5.21}$$

Our goal is to find initial data so that (5.15) holds, we shall pick

$$x^{(0)} = \bar{x},\ \delta_k^{(0)} = \cdots = \delta_K^{(0)} = 0,\ \text{and}\ (\delta_1^{(0)}, \ldots, \delta_{k-1}^{(0)}) = \bar{\omega}^{[t]} \tag{5.22}$$

for some small enough t. The remainder of the proof argues why this strategy works.

Step 2. Obviously, for t small enough

$$V_{\bar{\omega}^{[t]}} \subset B_\varepsilon(\mathbf{0}). \tag{5.23}$$

For such t, assume the choice of initial data as in (5.22), and denote the iterates by $x^{[t,j]}$ and $\delta^{[t,j]}$. By (5.23) and (5.21), we have

$$x^{[t,j]} \in U_{\bar{x}}, \qquad \text{for all } j. \tag{5.24}$$

By the Taylor expansion of Ψ at $(x^{[t,j]}, 0)$, together with Lemma 2.2 and the order $k-1$ differential proximity condition, we have the following for $\ell = 1, \ldots, k-1$:

$$\begin{aligned} \delta_\ell^{[t,j+1]} &= \Psi_\ell(x^{[t,j]}, \delta^{[t,j]}) \\ &= \frac{1}{2^\ell}\delta_\ell^{[t,j]} + \underbrace{\sum_{\substack{|\nu|=1 \\ \text{weight}(\nu)\geq \ell+1}}^{\ell} \frac{D^\nu \Psi_\ell|_{(x^{[t,j]},0)}}{\nu!}(\delta^{[t,j]})^\nu + \sum_{|\nu|=\ell+1} \frac{D^\nu \Psi_\ell|_{(x^{[t,j]},\tau\delta^{[t,j]})}}{\nu!}(\delta^{[t,j]})^\nu}_{=: y_\ell^{[t,j]}}, \end{aligned} \tag{5.25}$$

where $\tau = \tau(\delta^{[t,j]}, \ell) \in [0,1]$. (Note: When $|\nu| = 1$ with $\nu_k = 1$, $\text{weight}(\nu) = k$. Lemma 2.2 says that the linear part of Ψ_ℓ has the diagonal term $\frac{1}{2^\ell}\delta_\ell$ and all other terms are of weight greater than ℓ.) By Corollary 4.17, when $\ell = k$, we have instead

$$\begin{aligned} \delta_k^{[t,j+1]} &= \Psi_k(x^{[t,j]}, \delta^{[t,j]}) \\ &= \frac{1}{2^k}\delta_k^{[t,j]} + \text{Reson}_{x^{[t,j]}}\big(\delta_1^{[t,j]}, \ldots, \delta_{k-1}^{[t,j]}\big) + \\ &\qquad \underbrace{\sum_{\substack{|\nu|=1 \\ \text{weight}(\nu)\geq k+1}}^{k} \frac{D^\nu \Psi_\ell|_{(x^{[t,j]},0)}}{\nu!}(\delta^{[t,j]})^\nu + \sum_{|\nu|=k+1} \frac{D^\nu \Psi_\ell|_{(x^{[t,j]},\tau\delta^{[t,j]})}}{\nu!}(\delta^{[t,j]})^\nu}_{=: y_k^{[t,j]}}. \end{aligned} \tag{5.26}$$

From previous result, we know $\|\delta_\ell^{[t,j]}\| = O(2^{-\ell j})$, this implies $\|y_k^{[t,j]}\| = O(2^{-(\ell+1)j})$, so (5.26) becomes

$$\begin{aligned} \delta_k^{[t,j+1]} &= \frac{1}{2^k}\delta_k^{[t,j]} + \text{Reson}_{x^{[t,j]}}\big(\delta_1^{[t,j]}, \ldots, \delta_{k-1}^{[t,j]}\big) + O(2^{-(\ell+1)j}) \\ &= \frac{1}{2^k}\delta_k^{[t,j]} + 2^{-kj}\text{Reson}_{x^{[t,j]}}\big(2^j\delta_1^{[t,j]}, \ldots, 2^{(k-1)j}\delta_{k-1}^{[t,j]}\big) + O(2^{-(\ell+1)j}). \end{aligned} \tag{5.27}$$

Since we have (5.24), if we can only show that

$$\big(2^j\delta_1^{[t,j]}, \ldots, 2^{(k-1)j}\delta_{k-1}^{[t,j]}\big) \in V_{\bar{\omega}^{[t]}} \tag{5.28}$$

for any small enough t, then by (5.20) the resonance term in (5.27) stays uniformly positive or uniformly negative, the desired lower bound (5.15) then follows from Lemma 5.3.

Step 3. It remains to prove (5.28). Its proof relies on Lemma 5.4 and Proposition 3.22. By Lemma 5.4, if we can show that, for each $\ell = 1, \dots, k-1$,

$$\|y_\ell^{[t,j]}\| \le C_t\, \mu^j, \text{ for some } \mu < 2^{-\ell} \text{ and } C_t = o(t^\ell), \tag{5.29}$$

then we have

$$\delta_\ell^{[t,j]} 2^{\ell j} \in \Big[\underbrace{\delta_\ell^{[t,0]}}_{=\bar{\omega}_\ell^{[t]}} \pm o(t^\ell)\Big].$$

Recall from (5.17)-(5.19) that the ℓ-th dimensional cross section of $V_{\bar{\omega}^{[t]}}$ is also centered at $\bar{\omega}_\ell^{[t]}$ but has a width proportional to t^ℓ. Therefore, for small enough t, the hyper-rectangle

$$R_t := \Big[\bar{\omega}_1^{[t]} \pm o(t^1)\Big] \times \Big[\bar{\omega}_2^{[t]} \pm o(t^2)\Big] \times \cdots \times \Big[\bar{\omega}_{k-1}^{[t]} \pm o(t^{k-1})\Big]$$

must be contained in $V_{\bar{\omega}^{[t]}}$. See also Figure 5.

Step 4. We now prove (5.29) for $\mu = 2^{-(\ell+1-0.01)}$ and $C_t = O(t^{\ell+1})$. Note that

$$\|y_\ell^{[t,j]}\| \le A \sum_{\text{weight}(\nu)\ge \ell+1} \|\delta_1^{[t,j]}\|^{\nu_1} \cdots \|\delta_K^{[t,j]}\|^{\nu_K}, \tag{5.30}$$

where the sum on the right-hand side involves only a finite number of ν's, and $A > 0$ exists as a bounded constant because all the relevant derivatives in (5.25) are continuous and are evaluated at a bounded neighborhood, thanks to (5.21).

By Proposition 3.22 (with the 'k' and 'α' in the statement of Proposition 3.22 being $k-1$ and 1 here, respectively)

$$\|\delta_\ell^{[t,j]}\| \le C_\epsilon \cdot \begin{cases} 2^{-(\ell-\epsilon)j} \sum_{\text{weight}(\eta)\ge \ell} \big\|\delta_1^{[t,0]}\big\|^{\eta_1} \cdots \big\|\delta_{k-1}^{[t,0]}\big\|^{\eta_{k-1}}, & \ell = 1, \dots, k-1; \\ 2^{-(k-\epsilon)j} \sum_{\text{weight}(\eta)\ge k} \big\|\delta_1^{[t,0]}\big\|^{\eta_1} \cdots \big\|\delta_{k-1}^{[t,0]}\big\|^{\eta_{k-1}}, & \ell \ge k, \end{cases} \tag{5.31}$$

where the sums on the right-hand side involve only a finite number of multi-indices η. Recall also that we choose our initial $\delta_\ell^{[t,0]}$ to be zero when $\ell \ge k$.

Applying this estimate to (5.30) gives

$$\|y_\ell^{[t,j]}\| \le A' 2^{-j(\ell+1-\varepsilon)} \sum_{\text{weight}(\nu)\ge \ell+1} \big\|\delta_1^{[t,0]}\big\|^{\nu_1} \cdots \big\|\delta_{k-1}^{[t,0]}\big\|^{\nu_{k-1}} \tag{5.32}$$

for some constant $A' > 0$ and some $\varepsilon > 0$ which can be made arbitrarily small when ϵ is chosen to be small enough. And again the sum on the right-hand side only involves a finite number of multi-indices $\nu = (\nu_1, \dots, \nu_{k-1})$ all with weight at least $\ell + 1$. Therefore, by our choice of initial data (5.22), the sum on the right-hand side of (5.32) decays with t as $O(t^{\ell+1})$. □

5.2. **Proof of (II): Super-convergence.** The second result [II] is more difficult to motivate. Assume for the moment that S is a linear scheme. As we discussed in Section 2 part [I] alone is insufficient to infer that S is not C^k smooth; but if we assume the additional condition that S is L_∞-stable, then the implication would hold. The condition in (II) implies that S is in some sense "close enough" to an L_∞-stable linear scheme for (I) to imply that S is not C^k.

A hint for finding a replacement for the stability condition in the nonlinear setting is given by the following consideration: If S_{lin} were interpolatory, then one could dispense with the stability condition: for since $(S^j_{\text{lin}}c)_k = F(2^{-j}k)$, by a standard result in approximation theory (e.g. [4]) we have the following estimate:

$$\|\Delta^r_{2^{-j}} F(x)\|_{L^\infty} \asymp \|\Delta^r(F|_{2^{-j}\mathbb{Z}})\|_{\ell^\infty} = \|\Delta^r S^j_{\text{lin}} c\|_{\ell^\infty}.$$

It is then intuitively clear why the converse of (2.4) does not hold for a general *non-interpolatory* scheme, unless we have a way to control on the discrepancy between the sizes of $\Delta^r F|_{2^{-j}\mathbb{Z}}$ and $\Delta^r S^j_{\text{lin}} c$. With that in mind, it seems plausible to replace the stability condition by a *rate of convergence* condition.

In fact, the following 'analog' of (2.5) holds for any (linear or nonlinear) subdivision scheme:

$$\begin{aligned} &\text{If } S \text{ is } C^m \text{ and } \sup_{i\in\mathbb{Z}} \left|2^{jm}(\Delta^m S^j \mathbf{x})_i - F^{(m)}(2^{-j}i)\right|_\infty \lesssim 2^{-j\alpha}, \text{ then} \\ &S \text{ is } C^{m,\alpha} \implies \|\Delta^{m+1} S^j c\|_\infty \lesssim 2^{-j(m+\alpha)}. \end{aligned} \tag{5.33}$$

The proof is easy: Since $F^{(m)}$ is Hölder-α smooth, $\sup_k \left|F^{(m)}\left(2^{-j}(k+1)\right) - F^{(m)}\left(2^{-j}k\right)\right| \lesssim 2^{-j\alpha}$. Combining this with the rate of convergence condition in (5.33) using the triangle inequality yields the estimate $\sup_k \left|2^{jm}(\Delta^m S^j \mathbf{x})_{k+1} - 2^{jm}(\Delta^m S^j \mathbf{x})_k\right| \lesssim 2^{-j\alpha}$, which is equivalent to the decay condition in (5.33).

As it stands, (5.33) will not suffice. For assume that we have established that $\|\Delta^{m+1} S^j c\|_\infty$ decays *slower* than $2^{-j(m+\alpha)}$. To show that S is *not* $C^{m,\alpha}$ smooth, then we have to independently establish the rate of convergence condition in (5.33). But this is hopeless already in the linear case, for the linear theory tells us that if S is not $C^{m,\alpha}$ smooth, the rate of convergence in (5.33) may not be attained.

We resolve this problem in Section 5.2, where we prove (5.2), a *super-convergence* condition that replaces the convergence condition in (5.33).

We begin with the following observation about linear subdivision schemes:

Lemma 5.34. *Let S_{lin} be any linear subdivisions scheme. Assume S_{lin} reproduces Π_1.*[7] *Then S_{lin} interpolates all linear polynomials in the sense that*

$$S_{\text{lin}}\Big((p(k+\tau))_{k\in\mathbb{Z}}\Big) = \Big(p(k/2+\tau/2)_{k\in\mathbb{Z}}\Big), \quad \tau := \frac{1}{2}\sum_k a_k k, \tag{5.35}$$

for all $p \in \Pi_1$.

Proof. Let $(a_k)_k$ be the mask of S_{lin}. Write $(a_e)_k = a_{2k}$ and $(a_o)_k = a_{2k+1}$. So

$$\widehat{a}(z) = \widehat{a_e}(z^2) + z\widehat{a_o}(z^2). \tag{5.36}$$

Since S_{lin} reproduces Π_1, its mask also satisfies $\widehat{a_e}(1) = \widehat{a_o}(1) = 1$ and $\hat{a}'(-1) = 0$. By differentiating (5.36) and setting $z = -1$, we have $2\widehat{a_e}'(1) - 2\widehat{a_o}'(1) = 1$, which is equivalent to $2\sum_k a_{2k}k - 2\sum_k a_{2k+1}k = 1$. So if τ given by the formula in (5.35), a easy calculation shows that we also have $2\sum_k a_{2k}k = \tau$ and $2\sum_k a_{2k+1}k = \tau - 1$.

[7] Here, S_{lin} reproduces Π_k means $S_{\text{lin}}(\Pi_k) \subset \Pi_k$, which is equivalent to the Fourier domain condition $\widehat{a}^{(\ell)}(-1) = 0$, $0 \le \ell \le k$ [1, Lemma 3.1], or, equivalently, the time domain condition $\sum_k a_{2k}\pi(k+1/2) = \sum_k a_{2k+1}\pi(k)$ for all $\pi \in \Pi_k$. Here $\widehat{a}^{(\ell)}(z) = \sum_k a_k z^{-k}$ is the symbol of the mask of S_{lin}.

It suffices to prove the lemma only when $p(x) = x$, so let $p_k = k + \tau$, then

$$(S_{\text{lin}} p)_{2\ell+\sigma} = \sum_k a_{2k+\sigma} p_{\ell-k} = \ell - \sum_k a_{2k+\sigma} k + \tau$$
$$= \left\{ \begin{array}{ll} \ell + \tau/2, & \sigma = 0 \\ \ell + \tau/2 + 1/2, & \sigma = 1 \end{array} \right\} = p((2\ell + \sigma)/2 + \tau/2),$$

as desired. □

Remark 5.37. Comparing both sides of the refinement equation $\phi(x) = \sum_k a_k \phi(2x - k)$ and integrating, one arrives at the identity

$$\tau = \frac{1}{2} \sum_k a_k k = \frac{\int x\phi(x)\,dx}{\int \phi(x)\,dx}.$$

For this reason, the number τ in (5.36) is called the *centroid* of the refinable function ϕ associated with the linear subdivision scheme; we also call τ the *centroid of the linear subdivision scheme* S_{lin}.

If S_{lin} has a primal (resp. dual) symmetry, i.e. $a_k = a_{-k}$ (resp. $a_k = a_{1-k}$), then $\tau = 0$ (resp. $\tau = 1/2$). In general, τ lies inside the interval where the mask (a_k) is supported, and we may center the mask by an appropriate shift so that $\tau \in [0, 1)$.

Lemma 5.38. *If S_{lin} is Π_1-reproducing with centroid τ, then there exists a constant $C > 0$ independent of $\mathbf{x} \in \ell^\infty(\mathbb{Z} \to \mathbb{R})$ such that*

$$\max_k \left| (S_{\text{lin}} \mathbf{x})_{2k} - \frac{\tau}{2} \mathbf{x}_{k-1} - \left(1 - \frac{\tau}{2}\right) \mathbf{x}_k \right| \leq C \|\Delta^2 \mathbf{x}\|_\infty, \tag{5.39a}$$

$$\max_k \left| (S_{\text{lin}} \mathbf{x})_{2k+1} - \left(\frac{1}{2} + \frac{\tau}{2}\right) \mathbf{x}_k - \left(\frac{1}{2} - \frac{\tau}{2}\right) \mathbf{x}_{k+1} \right| \leq C \|\Delta^2 \mathbf{x}\|_\infty, \tag{5.39b}$$

$$\max_k \left| (1 - \tau)(S_{\text{lin}} \mathbf{x})_{2k} + \tau (S_{\text{lin}} \mathbf{x})_{2k+1} - \mathbf{x}_k \right| \leq C \|\Delta^2 \mathbf{x}\|_\infty. \tag{5.39c}$$

Proof. The proof of this lemma is based on a familiar fact: a finitely supported filter v annihilates all polynomial sequences p of degree not exceeding d, i.e. $v * p = 0$, for all $p \in \Pi_d$, if and only if $\hat{v}(z) = \hat{w}(z)(1 - z)^{d+1}$ for another finite supported filter w. In this case, $v * \mathbf{x} = w * \Delta^{d+1} \mathbf{x}$ for any sequence $\mathbf{x}$. Below, we use this fact for $d = 1$.

Note that each sequence on the left-hand side of (5.39a)-(5.39c) is the convolution of $\mathbf{x}$ with some finitely supported sequence v, therefore to prove the lemma it suffices to show that the left-hand sides all vanish when $\mathbf{x}$ is a sequence sampled from any linear polynomial. Here, it does not matter if we sample at $\mathbb{Z}$ or at $\mathbb{Z} + \tau$ as $p(k + \tau)$ and $p(k)$ differ by a constant when p is a linear polynomial. So it suffices to assume $x_k = p(k + \tau)$ for $p \in \Pi_1$. Now (5.39c) follows from Lemma 5.34. For (5.39a)-(5.39b), note that the two sequences on the left-hand sides are exactly $(S_{\text{lin}} \mathbf{x})_{2k+\sigma} - (S_\tau \mathbf{x})_{2k+\sigma}$, $\sigma = 0, 1$, where S_τ is the subdivision scheme with the mask $[1/2 - \tau/2, 1 - \tau/2, 1/2 + \tau/2, \tau/2]$ (supported at $-1, \ldots, 2$). Since S_τ is Π_1-reproducing with the same centroid τ as S_{lin}, again by Lemma 5.34 the left-hand sides of (5.39a)-(5.39b) vanish when $\mathbf{x}$ is sampled from a linear polynomial. □

The next theorem is a consequence of Lemma 5.38.

Theorem 5.40 (super-convergence: linear case). *Let $S_{\rm lin}$ be a stable Π_1-reproducing linear subdivision scheme. For any bounded initial sequence $\mathbf{x}$, let f_j be the piecewise linear function that interpolates $(S^j_{\rm lin}\mathbf{x})_k$ at $2^{-j}(k+\tau)$, where τ is the centroid of $S_{\rm lin}$.*

(a) If $S_{\rm lin}$ is $C^{1,\alpha}$, $\alpha\in(0,1]$, then

$$\|f_j - f_{j+1}\|_\infty \le C2^{-(1+\alpha)j}. \tag{5.41}$$

(b) If $S_{\rm lin}$ is in the Zgymund class Λ_*,[8] then

$$\|f_j - f_{j+1}\|_\infty \le C2^{-j}. \tag{5.42}$$

Remark 5.43. A standard result from linear subdivision theory says that if $S_{\rm lin}$ is $C^{0,1}$ smooth then the rate of convergence (5.42) holds. Since $\Lambda_* \supsetneqq C^{0,1}$, the rate of convergence (5.42) is faster than expected; the special interpolation based on the centroid τ is crucial for such a super-convergence. In fact, one should only expect the slower $O(j2^{-j})$ rate of convergence if we define f_j based on any $\tau\neq$ the centroid. A similar comment applies to part (a): without using the centroid, the rate of convergence is only $O(2^{-j})$ regardless of the value of $\alpha\in(0,1]$; with the centroid, the rate of convergence can go as fast as $O(2^{-2j})$ when $\alpha=1$.

If $S_{\rm lin}$ is C^{m-1} smooth, hence also reproduces Π_{m-1}, we always have a derived subdivision scheme $S^{[m]}_{\rm lin}$ which satisfies:

$$2^m\Delta^m S_{\rm lin} = S^{[m]}_{\rm lin}\Delta^m. \tag{5.44}$$

By applying Theorem 5.40 to $S^{[m]}_{\rm lin}$ we have the following generalization.

Corollary 5.45 (super-convergence: linear case). *Let $m\ge 0$ be an integer and $S_{\rm lin}$ be a stable Π_m-reproducing linear subdivision scheme. For any bounded initial sequence $\mathbf{x}$, let f_j be the piecewise linear function that interpolates $2^{mj}(\Delta^m S^j_{\rm lin}\mathbf{x})_k$ at $2^{-j}(k+\tau)$, where τ is centroid of the derived scheme $S^{[m]}_{\rm lin}$ above.*

(a) If $S_{\rm lin}$ is $C^{m+1,\alpha}$, $\alpha\in(0,1]$, then

$$\|f_j - f_{j+1}\|_\infty \le C2^{-(1+\alpha)j}. \tag{5.46}$$

(b) If $S_{\rm lin}$ is in the Zgymund class Λ^{m+1}_*, then

$$\|f_j - f_{j+1}\|_\infty \le C2^{-j}. \tag{5.47}$$

Theorem 5.48 (super-convergence: nonlinear case). *Let m be an integer and $S_{\rm lin}$ be a stable $C^{m+1,\alpha}$ smooth subdivision scheme. Let S be a subdivision scheme and f_j be the piecewise linear function that interpolates $2^{mj}(\Delta^m S^j\mathbf{x})_k$ at $2^{-j}(k+\tau)$, where τ is the centroid of $S^{[m]}_{\rm lin}$.*

[8] In this context, it means $\|\Delta^2 S^j_{\rm lin}x\|_\infty = O(2^{-j})$. In general, the Zgymund class [33] is the space of bounded functions which satisfy $\sup_x|\Delta^2_h f(x)| = O(h)$. In contrast, functions in $C^{0,1}$ (=Lip1) satisfy $\sup_x|\Delta_h f(x)| = O(h)$. It is well-known (e.g. [33, 17]) that $\Lambda_*\supsetneqq \text{Lip1}\supsetneqq C^1$. Similarly, Λ^{m+1}_* is the space of bounded functions with m-th derivatives in Λ_*; we have $C^m\supsetneqq\Lambda^{m+1}_*\supsetneqq C^{m,1}\supsetneqq C^{m+1}$.

(a) If S and $S_{\rm lin}$ satisfy the weak order $m+1$ proximity condition, then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\|f_j - f_{j+1}\|_\infty \leq C_\epsilon 2^{-(1+\alpha-\epsilon)j}. \tag{5.49}$$

(b) If S and $S_{\rm lin}$ satisfy the weak order m proximity condition, then

$$\|f_j - f_{j+1}\|_\infty \leq C 2^{-j}. \tag{5.50}$$

(In each case, the Cauchy sequence f_j converges to the m-th derivative of the C^m smooth limit function corresponding to the subdivision data $S^j\mathbf{x}$.)

Remark 5.51. Although the two parts of the theorem are similar, part (b) of this theorem is more relevant to (the necessity part of) the main result Theorem 1.19. When S and $S_{\rm lin}$ satisfy the order m but *not* the order $m+1$ proximity condition, then a *resonance effect* suggests that S is shy of $C^{m,1}$ smooth (Section 5.1), so from the experience in the linear theory one should not expect the rate of convergence (5.50) had we not used the more accurate method of interpolation based on the centroid. In analogy with part (b) of Theorem 5.40, one should expect the slower $O(j2^{-j})$ rate of convergence if we define f_j with τ not equal to the centroid.

Proof. We first prove part (b). The proof of (a) is similar, except that a few asymptotic terms become dependent on the assumed Hölder exponent α in $S_{\rm lin}$. (The α will appear in the guise of $\beta := \alpha - \epsilon$, where $\epsilon > 0$ can be arbitrarily small.) We shall see how the proof of (b) can be modified in order to prove part (a).

Step 1. Let S be a non-interpolatory subdivision scheme and $\mathbf{x}$ be a bounded sequence in $\mathbb{R}^n$. Since S and $S_{\rm lin}$ satisfy the order m proximity condition and $S_{\rm lin}$ is assumed to be stable, by the analysis in [28, Section 2], the divided differences $2^{j\ell}\Delta^\ell S^j\mathbf{x}$ converge to $\phi^{(\ell)}$ for each $\ell = 1, \ldots, m$. (So, in particular, S inherits the C^m smoothness from $S_{\rm lin}$.) Therefore, $\|2^{j\ell}\Delta^\ell S^j\mathbf{x}\|_\infty = O(1)$, or

$$\|\Delta^\ell S^j\mathbf{x}\|_\infty = O(2^{-j\ell}), \quad \ell = 1, \ldots, m. \tag{5.52}$$

The rest of this part of the proof finds a tight estimate for $\|\Delta^{m+2}S^j\mathbf{x}\|_\infty$. (See (5.61).)

Since $S_{\rm lin}$ is $C^{m+1,\alpha}$ smooth, there exists $C_0 > 0$ such that

$$\|\Delta^{m+2}S^j_{\rm lin}\mathbf{x}\|_\infty \leq C_0 2^{-(m+1+\alpha)j}\|\Delta^{m+2}\mathbf{x}\|_\infty. \tag{5.53}$$

Let $\beta \in (0, \alpha)$. Then there exists $L \in \mathbb{N}$ such that $C_0 \leq 2^{(\alpha-\beta)L}$. Hence,

$$C_0 2^{-(m+1+\alpha)L} \leq 2^{-(m+1+\beta)L}. \tag{5.54}$$

It follows from [28, Lemma A.3] that S^ℓ and $S^\ell_{\rm lin}$ also satisfy an order m proximity condition for each $\ell = 1, 2, \cdots, L$, i.e.

$$|\Delta^{m-1}S^\ell\mathbf{x} - \Delta^{m-1}S^\ell_{\rm lin}\mathbf{x}|_\infty \lesssim \Omega_m(\mathbf{x}), \quad \ell = 1, 2, \cdots, L. \tag{5.55}$$

Hence, for $\ell = 1, 2, \cdots, L$,

$$\|\Delta^{m+2}S^\ell\mathbf{x} - \Delta^{m+2}S^\ell_{\rm lin}\mathbf{x}\|_\infty \leq 8\|\Delta^{m-1}S^\ell\mathbf{x} - \Delta^{m-1}S^\ell_{\rm lin}\mathbf{x}\|_\infty \lesssim \Omega_m(\mathbf{x}). \tag{5.56}$$

It follows that

$$\|\Delta^{m+2}S^\ell\mathbf{x}\|_\infty \leq \|\Delta^{m+2}S^\ell_{\rm lin}\mathbf{x}\|_\infty + O(\Omega_m(\mathbf{x})) \quad \ell = 1, 2, \cdots, L. \tag{5.57}$$

Replacing $\mathbf{x}$ by $S^{L(j-1)}\mathbf{x}$, we obtain

$$\begin{aligned}\|\Delta^{m+2}S^{L(j-1)+\ell}\mathbf{x}\|_\infty &\leq \|\Delta^{m+2}S^{\ell}_{\text{lin}}S^{L(j-1)}\mathbf{x}\|_\infty + O\Big(\Omega_m\big(S^{L(j-1)}\mathbf{x}\big)\Big)\\ &\leq C_0 2^{-(m+1+\alpha)\ell}\|\Delta^{m+2}S^{L(j-1)}\mathbf{x}\|_\infty + O(2^{-(m+1)L(j-1)}).\end{aligned} \tag{5.58}$$

When $\ell = L$, we have, by (5.54),

$$\underbrace{\|\Delta^{m+2}S^{Lj}\mathbf{x}\|_\infty}_{=:D^j} \leq \underbrace{2^{-(m+1+\beta)L}}_{=:\rho}\underbrace{\|\Delta^{m+2}S^{L(j-1)}\mathbf{x}\|_\infty}_{D^{j-1}} + O((\underbrace{2^{-(m+1)L}}_{=:r})^{(j-1)}). \tag{5.59}$$

Iterating this inequality j times yields:

$$\begin{aligned}\|\Delta^{m+2}S^{Lj}\mathbf{x}\|_\infty = D^j &\leq \rho^j D^0 + O\Big(\sum_{i=0}^{j-1}\rho^{j-1-i}r^i\Big)\\ &= \rho^j D^0 + O\Big(\rho^{j-1}\sum_{i=0}^{j-1}(\frac{r}{\rho})^i\Big)\\ &= \rho^j D^0 + O\Big(\rho^{j-1}2^{\beta L(j-1)}\Big) = O(2^{-(m+1)Lj}).\end{aligned} \tag{5.60}$$

(**Note**: In the last step above, we have $r/\rho = 2^{\beta L} > 1$. In this case the magnitude of β does not affect the asymptotic behavior. As we shall see, the situation is different in Part (a).)

For any $J \in \mathbb{N}$, write $J = L(j-1) + \ell$, then by (5.58) and (5.60), we have

$$\|\Delta^{m+2}S^J\mathbf{x}\|_\infty \lesssim 2^{-(m+1)J}. \tag{5.61}$$

Step 2. Next, we need a trivial fact: two linear functions $l_1(x)$, $l_2(x)$ defined on an interval $[a, b]$ have the largest difference at either one of the two boundary points, i.e. $\max_{a\leq x\leq b}|l_1(x) - l_2(x)| \leq \max(|l_1(a) - l_2(a)|, |l_1(b) - l_2(b)|)$. Therefore, to prove (5.50), it suffices to prove

$$\max_k \Big\|2^{m(j+1)}(\Delta^m S^{j+1}\mathbf{x})_{2k} - \frac{\tau}{2}2^{mj}(\Delta^m S^j\mathbf{x})_{k-1} - (1-\frac{\tau}{2})2^{mj}(\Delta^m S^j\mathbf{x})_k\Big\| \lesssim 2^{-j},$$

$$\max_k \Big\|2^{m(j+1)}(\Delta^m S^{j+1}\mathbf{x})_{2k+1} - (\frac{1}{2}+\frac{\tau}{2})2^{mj}(\Delta^m S^j\mathbf{x})_k - (\frac{1}{2}-\frac{\tau}{2})2^{mj}(\Delta^m S^j\mathbf{x})_{k+1}\Big\| \lesssim 2^{-j},$$

$$\max_k \Big\|(1-\tau)2^{m(j+1)}(\Delta^m S^{j+1}\mathbf{x})_{2k} + \tau 2^{m(j+1)}(\Delta^m S^{j+1}\mathbf{x})_{2k+1} - 2^{mj}(\Delta^m S^j\mathbf{x})_k\Big\| \lesssim 2^{-j},$$

which are equivalent to

$$\max_k \Big\|2^m(\Delta^m S^{j+1}\mathbf{x})_{2k} - \frac{\tau}{2}(\Delta^m S^j\mathbf{x})_{k-1} - (1-\frac{\tau}{2})(\Delta^m S^j\mathbf{x})_k\Big\| \lesssim 2^{-(m+1)j}, \tag{5.62a}$$

$$\max_k \Big\|2^m(\Delta^m S^{j+1}\mathbf{x})_{2k+1} - (\frac{1}{2}+\frac{\tau}{2})(\Delta^m S^j\mathbf{x})_k - (\frac{1}{2}+\frac{\tau}{2})(\Delta^m S^j\mathbf{x})_{k+1}\Big\| \lesssim 2^{-(m+1)j}, \tag{5.62b}$$

$$\max_k \Big\|(1-\tau)2^m(\Delta^m S^{j+1}\mathbf{x})_{2k} + \tau 2^m(\Delta^m S^{j+1}\mathbf{x})_{2k+1} - (\Delta^m S^j\mathbf{x})_k\Big\| \lesssim 2^{-(m+1)j}. \tag{5.62c}$$

Note that $S^{[m]}_{\text{lin}}$ as defined by (5.44) reproduces Π_1, as S_{lin} is assumed to be C^{m+1} smooth and hence reproduces Π_{m+1}. This means we can apply the estimate (5.39a) in Lemma 5.38 with S_{lin} replaced by $S_{\rm lin}^{[m]}$ and $\mathbf{x}$ replaced by the sequence $\Delta^m S^j\mathbf{x}$ to get

$$\begin{aligned}&\max_k \big\|\big(\overbrace{S_{\rm lin}^{[m]}\Delta^m}^{\overset{(5.44)}{=}2^m\Delta^m S_{\rm lin}} S^j\mathbf{x})_{2k} - \frac{\tau}{2}(\Delta^m S^j\mathbf{x})_{k-1} - (1-\frac{\tau}{2})(\Delta^m S^j\mathbf{x})_k\big\| \\ \lesssim &\|\Delta^2\Delta^m S^j\mathbf{x}\|_\infty = \|\Delta^{m+2}S^j\mathbf{x}\|_\infty \overset{(5.61)}{\lesssim} 2^{-(m+1)j},\end{aligned}$$

where τ is the centroid of $S_{\rm lin}^{[m]}$. This is almost what we want in (5.62a), except that the appearance of $S_{\rm lin}$ on the left-hand side must be replaced by the nonlinear S. The order m proximity condition and (5.52) do the job, as they imply that

$$\|\Delta^m S^{j+1}\mathbf{x} - \Delta^m S_{\rm lin}S^j\mathbf{x}\|_\infty = \|\Delta^m S^{j+1}\mathbf{x} - \Delta^m S_{\rm lin}S^j\mathbf{x}\|_\infty \lesssim 2^{-(m+1)j}. \tag{5.63}$$

This, together with the triangle inequality, gives (5.62a):

$$\begin{aligned}&\max_k \|2^m(\Delta^m S^{j+1}\mathbf{x})_{2k} - \frac{\tau}{2}(\Delta^m S^j\mathbf{x})_{k-1} - (1-\frac{\tau}{2})(\Delta^m S^j\mathbf{x})_k\| \\ \leq &2^m\|\Delta^m SS^j\mathbf{x} - \Delta^m S_{\rm lin}S^j\mathbf{x}\|_\infty + \max_k \Big\|2^m(\Delta^m S_{\rm lin}S^j\mathbf{x})_{2k} - \frac{\tau}{2}(\Delta^m S^j\mathbf{x})_{k-1} - (1-\frac{\tau}{2})(\Delta^m S^j\mathbf{x})_k\Big\| \\ = &O(2^{-(m+1)j}) + O(2^{-(m+1)j}).\end{aligned}$$

The proofs of (5.62b) and (5.62c) are completely analogous.

We now prove part (a). If we assume one more order of proximity condition between S and $S_{\rm lin}$, then (5.52) holds for $\ell = 1, \ldots, m+1$. Since the smoothness assumption on $S_{\rm lin}$ is unchanged, (5.53)-(5.54) stay the same. With the additional order of proximity condition, (5.55)-(5.57) have to be changed accordingly, and (5.57) becomes:

$$\|\Delta^{m+2}S^\ell\mathbf{x}\|_\infty \leq \|\Delta^{m+2}S_{\rm lin}^\ell\mathbf{x}\|_\infty + O(\Omega_{m+1}(\mathbf{x})) \quad \ell = 1, 2, \cdots, L. \tag{5.64}$$

Then (5.65) becomes

$$\|\Delta^{m+2}S^{L(j-1)+\ell}\mathbf{x}\|_\infty \leq C_0 2^{-(m+1+\alpha)\ell}\|\Delta^{m+2}S^{L(j-1)}\mathbf{x}\|_\infty + O(2^{-(m+2)L(j-1)}). \tag{5.65}$$

And (5.59) is of the same form $D^j \leq \rho D^{j-1} + O(r^{j-1})$ with the same D^j and $\rho = 2^{-(m+1+\beta)L}$ but now r becomes $r = 2^{-(m+2)L}$. In this case

$$\frac{r}{\rho} = 2^{-(1-\beta)L} < 1.$$

As such, the estimate in (5.60) is changed as follows:

$$\|\Delta^{m+2}S^{Lj}\mathbf{x}\|_\infty \leq \rho^j D^0 + O\Big(\rho^{j-1}\sum_{i=0}^{j-1}(\frac{r}{\rho})^i\Big) = \rho^j D^0 + O(\rho^{j-1}) = O(2^{-(m+1+\beta)Lj}). \tag{5.66}$$

Then by (5.65) and (5.66), (5.61) is changed to

$$\|\Delta^{m+2}S^J\mathbf{x}\|_\infty \lesssim 2^{-(m+1+\beta)J}. \tag{5.67}$$

In order to adapt Step 2 of the proof of part (b) to part (a), we need only change every appearance of 2^{-j} to $2^{-(1+\beta)j}$ and every appearance of $2^{-(m+1)j}$ to $2^{-(m+1+\beta)j}$. Finally, recall that β can be chosen to be arbitrarily close to, but smaller than, α. This completes the proof of (a). $\square$

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