ON A NEW PROXIMITY CONDITION FOR MANIFOLD-VALUED SUBDIVISION SCHEMES

TOM DUCHAMP, GANG XIE, AND THOMAS YU

Abstract. An open theoretical problem in the study of subdivision algorithms for approximation of manifold-valued data has been to give necessary and sufficient conditions for a manifold-valued subdivision scheme, based on a linear subdivision scheme, to share the same regularity as the linear scheme. This is called the *smoothness equivalence problem*. In a companion paper, the authors introduced a *differential proximity condition* which solves the smoothness equivalence problem. In this paper, we review this condition, comment on a few of its unanticipated features, and as an application, show that the single base-point log-exp scheme suffers from an intricate breakdown of smoothness equivalence. We also show that the differential proximity condition is coordinate independent, even when the linear scheme is not assumed to possess the relevant smoothness.

Date: April 9, 2014.

2000 *Mathematics Subject Classification.* 41A25, 26B05, 22E05, 68U05.

Key words and phrases. Differential Proximity Condition, Nonlinear subdivision, Manifold, Curvature, Symmetry, Super-Convergence, Nonlinear Dynamical System.

Tom Duchamp gratefully acknowledges the support and hospitality provided by the IMA during his visit from April to June 2011, when part of the work in this article was completed, as well as travel support through the PIMS CRG on Applied and Computational Harmonic Analysis.

Gang Xie's research was supported by the Fundamental Research Funds for the Central Universities and the National, Natural Science Foundation of China (No.11101146).

Thomas Yu's research was partially supported by the National Science Foundation grants DMS 0915068 and DMS 1115915, as well as a fellowship offered by the Louis and Bessie Stein family.

1. Introduction

In recent years, manifold-valued data has become ubiquitous; the configuration spaces of robots and space of anisotropic diffusion tensors are but two examples. Although manifolds, by definition, can be locally parametrized by points in Euclidean space, such a local parametric representation is insufficient when the topology of the underlying space is non-trivial (e.g. configuration space), and even in the case of trivial topology (anisotropic diffusion) it is desirable to respect the natural symmetry and metric structure of the underlying manifold. For these reasons, genuinely nonlinear, differential geometric, approximation methods have come to play an important role.

Recently, several research groups [12, 11, 13, 14], [6, 5, 4, 8, 7], [17, 16, 18, 17, 21, 20, 3, 19] have studied subdivision methods for manifold-valued data. Roughly speaking, a subdivision method takes as input coarse scale data and recursively generates data at successively finer scales with the hope that in the limit a function with desired regularity properties is obtained. Such algorithms have attracted the interest of applied analysts not only because of their intrinsic beauty, but also because of their connection with wavelet-like representations. In this context, various approximation-theoretic questions come to mind, such as: How much regularity does the limit function possess? At what rate does it approximate the underlying function from which the coarse data originates?

A number of different subdivision schemes for manifold-valued data were introduced in the above references: some exploit the exponential map, others a retraction map, and some the Karcher mean, yet others are based on an embedding of the manifold into Euclidean space. But in all cases, the subdivision method is modeled on an underlying linear subdivision scheme. It is therefore natural to seek conditions under which the limit function of a manifold-valued subdivision method enjoys the same limit properties as the limit function of underlying linear subdivision scheme. This is called the *smoothness equivalence problem*. We and others [16, 12, 18, 17, 21, 6, 5, 4, 13, 3], have introduced various *proximity conditions* that are sufficient for a manifold-valued scheme to have the smoothness equivalence property. Although numerical evidence for the necessity of these proximity conditions were given in [17, 21, 2], necessity has remained an open problem.

In a companion paper [2], we present a complete solution of the smoothness equivalence problem in terms of a new proximity condition, which we call the *differential proximity condition*. Here, we review this condition, comment on a few of its unanticipated features, and as an application, we show why the single base-point log-exp scheme suffers from an intricate breakdown of smoothness equivalence. We also prove that the differential proximity condition is coordinate independent. The coordinate independence result established in Section 4 is stronger than what would follow immediately from the main result in [2].

2. Smooth Compatibility and the Differential Proximity Condition

Let M be a differentiable manifold of dimension n. A map $S : \ell(\mathbb{Z} \to M) \to \ell(\mathbb{Z} \to M)$ is called a **subdivision scheme on** M if it is of the form

$$(S\mathbf{x})_{2i+\sigma} = q_\sigma(x_{i-m_\sigma}, \cdots, x_{i-m_\sigma+L_\sigma}), \quad \sigma = 0, 1,\ i \in \mathbb{Z}, \tag{2.1}$$

where $L_\sigma, m_\sigma \in \mathbb{Z}$, $L_\sigma > 1$, and q_σ are continuous maps

$$q_\sigma : \underbrace{M \times \cdots \times M}_{L_\sigma + 1 \text{ copies}} \to M, \quad \sigma = 0, 1, \tag{2.2}$$

defined in a neighborhood of the hyper-diagonal of $M \times \cdots \times M$ and satisfying the condition

$$q_\sigma(x, \ldots, x) = x\,. \tag{2.3}$$

The maps q_0, q_1 are usually referred to as the **even and odd rules** of the subdivision scheme S. In general, q_σ are only defined in a neighborhood of the hyper-diagonal, and therefore S is only defined for locally sufficiently dense sequences. We call L_σ the *locality factors* and m_σ the *phase factors* of the subdivision scheme S. The above definition was used, for example, in [20] and [15].

We now impose additional conditions on S.

Definition 1. Let S be a subdivision scheme on M. Let S_{lin} be a linear subdivision scheme with the same phase and locality factors as S and let $q_{\text{lin},\sigma}$, $\sigma = 0, 1$, be the (linear) maps associated with S_{lin}, as in (2.1). We say that S is **smoothly compatible**[1] **with** S_{lin} if

(a) q_0 and q_1 are (C^∞) smooth maps, and
(b) for any $x \in M$, $dq_\sigma|_{(x,\ldots,x)} : T_xM \times \cdots \times T_xM \to T_xM$ satisfies the condition

$$dq_\sigma|_{(x,\ldots,x)}(X_0, \ldots, X_{L_\sigma}) = q_{\text{lin},\sigma}(X_0, \ldots, X_{L_\sigma}), \quad \sigma = 0, 1.$$

Remark 2. The maps $q_{\text{lin},\sigma}$, $\sigma = 0, 1$, are the even and odd rules of S_{lin}. The compatibility condition in Definition 1 is satisfied by all the manifold-valued data subdivision schemes seen in the literature [9, 16, 18, 17, 21, 12, 6, 5, 4, 13].

Assume that S satisfies the compatibility condition in Definition 1. Our differential proximity condition is defined in terms of a finite-dimensional map Q. From Equations (2.1) and (2.2), it follows that there is a unique integer K such that any $K + 1$ consecutive entries in any (dense enough) sequence $\mathbf{x}$ determines **exactly** $K + 1$, and no more, consecutive entries in $S\mathbf{x}$. We may call $K + 1$ the size of a **minimal invariant neighborhood** of S. For any linear C^k subdivision scheme,

$$K \geq k,$$

with equality attained by the C^k, degree $k + 1$, B-spline subdivision scheme (See Figure 1). It follows that there is a map

$$Q : U \longrightarrow U \subset \underbrace{M \times \cdots \times M}_{K + 1 \text{ copies}}, \tag{2.4}$$

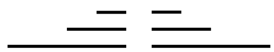
for U a sufficiently small open neighborhood of the hyper-diagonal, such that if $\mathbf{y} = S\mathbf{x}$, then

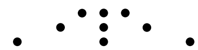
$$Q([\mathbf{x}_i, \ldots, \mathbf{x}_{i+K}]) = [\mathbf{y}_{2i+s}, \ldots, \mathbf{y}_{2i+s+K}], \tag{2.5}$$

for all i. The integer s, called a *shift factor*, is a constant independent of i but dependent on the phase factors of S. A basic property of S is that when the input sequence $\mathbf{x}$ is shifted by one entry, then the subdivided sequence $\mathbf{y}$ is shifted by two entries. This property is also reflected in equation (2.5).

[1] In [7, Definition 3.5], Grohs gives a similar compatibility condition.

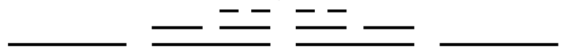
$k = 1 = K$ $\qquad\qquad$ $k = 2 = K$





$$Q(x_0, x_1) = \begin{bmatrix} q_1(x_0, x_1) \\ q_0(x_0, x_1) \end{bmatrix} \quad Q(x_0, x_1, x_2) = \begin{bmatrix} q_1(x_0, x_1) \\ q_0(x_0, x_1, x_2) \\ q_1(x_1, x_2) \end{bmatrix}$$

$k = 3 = K$



$$Q(x_0, x_1, x_2, x_3) = \begin{bmatrix} q_0(x_0, x_1, x_2) \\ q_1(x_0, x_1, x_2) \\ q_0(x_1, x_2, x_3) \\ q_1(x_1, x_2, x_3) \end{bmatrix}$$

FIGURE 1. If S is the symmetric C^k (degree $k+1$) B-Spline subdivision scheme, the corresponding map Q has a minimal invariant neighborhood of size $K + 1 = k + 1$. The figure shows two subdivision steps starting from $k + 1$ entries of the initial sequence. (Dots and intervals are used only because of the primal and dual symmetries in the B-Spline subdivision schemes for odd and even k. The symmetry properties, however, play no role here.)

The compatibility condition implies that

$$dQ|_{(x,\ldots,x)} = Q_{\rm lin}, \quad \forall\, x \in M, \tag{2.6}$$

where $Q_{\rm lin} : T_xM \times \cdots \times T_xM \to T_xM \times \cdots \times T_xM$ is the corresponding linear self-map defined by the maps $q_{{\rm lin},\sigma}$ in the compatibility condition.

We shall define our new order k proximity condition based on the higher order behavior of the map Q. At this point, we work in local coordinates on M. Let Q be the map $Q(x_0, x_1, \ldots, x_K)$ expressed in local coordinates around $x_0 \in M$, and define $\Psi : \mathbb{R}^n \times \cdots \times \mathbb{R}^n (K + 1 \text{ copies}) \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ by

$$\Psi := \nabla \circ Q \circ \Sigma, \tag{2.7}$$

where $\nabla, \Sigma = \nabla^{-1} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ are the linear maps defined by the correspondence

$$(x_0, x_1, \ldots, x_K) \underset{\Sigma}{\overset{\nabla}{\rightleftarrows}} (\delta_0 = x_0, \delta_1, \ldots, \delta_K) \tag{2.8}$$

where $\delta_k :=$ k-th order difference of $x_0, x_1, \ldots, x_k$, so

$$\delta_k = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} x_\ell, \text{ and } x_k = \sum_{\ell=0}^{k} \binom{k}{\ell} \delta_\ell\,. \tag{2.9}$$

Note that Ψ is only defined in a neighborhood of $(x_0, 0, \ldots, 0)$. (Here, by abuse of notation, we identify points in M with the corresponding points in $\mathbb{R}^n$ under the given coordinate chart.) We write

$$\Psi = (\Psi_0, \Psi_1, \ldots, \Psi_K), \quad \Psi_\ell : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n\,,$$

when referring to the different components of Ψ.

We remark that Equation (2.6), together with linearity of Σ and ∇, implies the identity

$$d\Psi|_{(x,0,\dots,0)} = \Psi_{\text{lin}} := \nabla \circ Q_{\text{lin}} \circ \Sigma, \quad \forall\, x, \tag{2.10}$$

Definition 3. Let S be a subdivision scheme on M smoothly compatible with S_{lin}. Let $k \geq 1$. We say that S and S_{lin} satisfy an **order k differential proximity condition** if for any $x_0 \in M$,

$$D^\nu \Psi_\ell|_{(x_0,0,\dots,0)} = 0, \;\text{ when } |\nu| \geq 2, \;\text{ weight}(\nu) := \sum j\nu_j \leq \ell, \quad \forall\, \ell = 1,\dots,k, \tag{2.11}$$

where $D^\nu \Psi_\ell$ denotes the derivative of Ψ_ℓ with respect to the multi-index $\nu = (\nu_1,\dots,\nu_K)$.

Remark 4. In above, $\nu = (\nu_1,\dots,\nu_K)$ does not have a 0-th component, so D^ν does not differentiate with respect to the 0-th argument. But since (2.11) has to hold for arbitrary x_0, then under the smooth compatibility assumption, condition (2.11) would be unaltered if we interpret ν as $(\nu_0,\nu_1,\dots,\nu_K)$.

Remark 5. In fact, the above condition is equivalent to the following seemingly stronger condition:

$$D^\nu \Psi_\ell|_{(x_0,0,\dots,0)} = 0, \;\text{ when } |\nu| \geq 2, \quad \text{weight}(\nu) \leq \begin{cases} \ell, & 1 \leq \ell \leq k \\ k, & \ell > k \end{cases}. \tag{2.12}$$

The proof, however, is rather technical as it relies on a major algebraic structure found in the proof of the sufficiency part of the following main result. See the sufficiency section of [2].

In Section 4 we need the following property of linear subdivision schemes:

Lemma 6. *If S_{lin} reproduces Π_k (=the space of polynomials of degree not exceeding k), then Ψ_{lin} has the block **upper triangular** form:*

$$\Psi_{\text{lin},\ell}(\delta_0,\delta_1,\dots,\delta_K) = \begin{cases} \frac{1}{2^\ell}\delta_\ell + \sum_{\ell'=\ell+1}^K U_{\ell,\ell'}\delta_{\ell'}, & \ell = 0,\dots,k \\ \sum_{\ell'=k+1}^K U_{\ell,\ell'}\delta_{\ell'}, & \ell = k+1,\dots,K \end{cases}, \tag{2.13}$$

where $U_{\ell,\ell'}$ are scalars *dependent only on the mask of S_{lin}. Moreover, if S_{lin} is C^k smooth, the spectral radius of the lower right block $[U_{\ell,\ell'}]_{\ell,\ell'=k+1,\dots,K}$ is strictly smaller than $1/2^k$.*

Remark 7. We may combine Lemma 6 with (2.11) to restate the differential proximity condition as:

$$D^\nu \Psi_\ell|_{(x_0,0,\dots,0)} = \begin{cases} \frac{1}{2^\ell}\text{id}, & |\nu| = 1 \text{ and weight}(\nu) = \ell, \\ 0, & |\nu| = 1 \text{ and weight}(\nu) < \ell, \text{ or} \\ & |\nu| \geq 2 \text{ and weight}(\nu) \leq \ell, \end{cases} \tag{2.14}$$

for $\ell = 1,\dots,k$.

In [2], we establish the following:

Theorem 8. *Let S be a subdivision scheme on a manifold smoothly compatible with a stable C^k smooth linear scheme S_{lin}. Then S is C^k smooth **if and only if** it satisfies the order k differential proximity condition.*

Unlike the compatibility condition, the differential proximity condition is expressed in local coordinates. A natural question is whether the latter condition is invariant under change of coordinates. For the original proximity conditions, the invariance question was answered in the affirmative in [20]. Armed with Theorem 8, we know that the order k differential proximity condition, being equivalent to the C^k smoothness of S, cannot be satisfied in one chart but not another, as the notion of smoothness is coordinate independent. In summary, we have:

Corollary 9. *If S is smoothly compatible with a stable C^k linear subdivision scheme $S_{\rm lin}$, then the order k differential proximity condition is invariant under change of coordinates.*

3. What's new?

The original proximity condition, used in our previous work, reads

$$\|\Delta^{j-1} S\mathbf{x} - \Delta^{j-1} S_{\rm lin}\mathbf{x}\|_\infty \le C\,\Omega_j(\mathbf{x}), \quad j = 1, \ldots, k, \tag{3.1}$$

where $\Omega_j(\mathbf{x}) := \sum_{\gamma\in\Gamma_j} \prod_{i=1}^{j} \|\Delta^i \mathbf{x}\|_\infty^{\gamma_i}$, $\Gamma_j := \Big\{\gamma = (\gamma_1, \cdots, \gamma_j)\,\Big|\, \gamma_i \in \mathbb{Z}^+,\ \sum_{i=1}^{j} i\,\gamma_i = j+1\Big\}$. It is well-known that this condition is sufficient condition for the C^k-equivalence property ([17, Theorem 2.4].) Moreover, years of usage of this condition (3.1) and numerical evidence suggests that it is also necessary.

This original proximity condition does not explicitly assume a compatibility condition between S and $S_{\rm lin}$, making it difficult to formulate a precise necessary condition. In our new formulation, we explicitly impose the smooth compatibility condition in Definition 1, which enables us to address the problem of necessity.

Our new formulation also addresses a perplexing aspect of condition (3.1). A careful inspection of the proof of [17, Theorem 2.4], shows that only the following proximity condition is needed[2]:

$$\|\Delta^{j} S\mathbf{x} - \Delta^{j} S_{\rm lin}\mathbf{x}\|_\infty \le C\,\Omega_j(\mathbf{x}), \quad j = 1, \ldots, k, \tag{3.2}$$

provided that we have already established C^0 regularity of S. We are thus faced with a dilemma: Despite the strong empirical evidence for the necessity of the proximity condition (3.1), it appears that it is unnecessarily strong!

A moment's thought suggests that the new proximity condition (2.11) is merely a differential version of the weaker condition (3.2). In fact, in all previous work a proximity condition is always established by a local Taylor expansion of $S\mathbf{x} - S_{\rm lin}\mathbf{x}$ (recall that subdivision schemes act locally). Consequently, the differential aspect of (2.11) is hardly anything new. But once the differential proximity condition is written in the form (2.11) (or in the equivalent form (2.14)), we see a natural interpretation: Viewing the map

$$Q : U \to U$$

as a discrete dynamical system, the proximity conditions can be interpreted in terms of the rate of approach of points in U to the hyper-diagonal, which is the fixed-point set of Q:

- Condition (2.14) then suggests that the linear term $2^{-\ell}\mathrm{id}$ is the dominant term, so, generically, the k-th order differences of the subdivision data *within any invariant neighborhood* (Figure 1) decays like $O(2^{-jk})$.

  If k is the first order at which the differential proximity condition fails, then there is a weight k term in the Taylor expansion of Ψ_k, such a nonlinear term is called a ***resonance term*** in the dynamical system literature, and the dynamical system interpretation would suggest that the k-th order differences of the subdivision data decays slower than $O(2^{-jk})$. More precisely, the presence of resonance slows down the decay to $O(j2^{-jk})$.

[2]Use $\|\Delta\mathbf{x}\|_\infty \le 2\|\mathbf{x}\|_\infty$ to see that (3.1) implies (3.2).

- This in turn suggests the necessity result. However, proving the lower bound result needed and tackling the lack of stability condition in the nonlinear subdivision theory are technically difficult. The former requires us to come up with a delicate argument to show that initial data exists so that the effect of resonance terms would not dissipate away in the course of iteration. The latter requires us to exploit a subtle super-convergence property.
- The same dynamical system interpretation suggests that our differential proximity condition may be too weak. For, unlike (3.1) or (3.2), it involves a fixed, although arbitrary, invariant neighborhood, and therefore does not appear to capture the expanding nature of a subdivision scheme.[3] Worse, the sufficiency part of the theorem concerns *establishing* the C^k-smoothness of the limiting function, and that would require one to analyze the decay rate of order $k+1$, not k, differences. If one examines Figure 1, one sees that a minimal invariant neighborhood may very well be too small to allow for the computation of any $k+1$ order difference. Fortunately, an unexpected algebraic structure we discovered in [2] not only makes the seemingly impossible mission of proving sufficiency possible, but also explains simultaneously why the apparent stronger-than-necessary proximity condition (3.1) always holds true whenever C^k equivalence holds.

4. Coordinate Independence

Corollary 9 suggests that there is an intrinsic, coordinate-free, reformulation of the differential proximity condition is waiting to be discovered. With this in mind, we establish here the following coordinate independence result.

Theorem 10. *If S is smoothly compatible with a Π_k reproducing linear subdivision scheme S_{lin}, then the order k differential proximity condition is invariant under change of coordinates.*

Notice that this result is stronger than Corollary 9, because a stable C^k linear subdivision scheme must reproduce Π_k, but the converse is far from being true.

Let $\chi(x) = \overline{x}$ be the change of coordinate map on M, and let $Q(x_0, x_1, \ldots, x_K)$ and $\overline{Q}(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_K)$ denote the expressions for map Q in these two coordinate systems. Writing

$$\chi_{\mathrm{vec}}(x_0, x_1, \ldots, x_K) := (\chi(x_0), \chi(x_1), \ldots, \chi(x_K)),$$

shows that $Q(x_0, x_1, \ldots, x_K)$ and $\overline{Q}(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_K)$ are related by the formula

$$\overline{Q} = \chi_{\mathrm{vec}} \circ Q \circ \chi_{\mathrm{vec}}^{-1}. \tag{4.1}$$

The map Ψ, by the definition (2.7), then takes the following forms in the two coordinate systems:

$$\Psi(\delta_0, \delta_1, \ldots, \delta_K) = \nabla \circ Q \circ \Sigma(\delta_0, \delta_1, \ldots, \delta_K),$$
$$\overline{\Psi}(\overline{\delta}_0, \overline{\delta}_1, \ldots, \overline{\delta}_K) = \nabla \circ \overline{Q} \circ \Sigma(\overline{\delta}_0, \overline{\delta}_1, \ldots, \overline{\delta}_K).$$

It then follows that a change of coordinates induces the following transformation rule for Ψ:

$$\overline{\Psi} = \nabla \circ \overline{Q} \circ \Sigma = \nabla \circ \chi_{\mathrm{vec}} \circ Q \circ \chi_{\mathrm{vec}}^{-1} \circ \Sigma = \underbrace{\nabla \circ \chi_{\mathrm{vec}} \circ \Sigma}_{=:\Xi} \circ \underbrace{\nabla \circ Q \circ \Sigma}_{=\Psi} \circ \underbrace{\nabla \circ \chi_{\mathrm{vec}}^{-1} \circ \Sigma}_{=\Xi^{-1}}. \tag{4.2}$$

[3]For instance, it is well-known from the linear theory that the spectral property of Ψ_{lin} alone is insufficient for characterizing the regularity property S_{lin}.

Proof. The proof proceeds in two steps:

Step 1. Notice the following structure of the Taylor expansion of $\Xi_\ell(\delta_0, \delta_1, \ldots, \delta_K)$ around a point $(\delta_0 = x_0, 0, \ldots, 0)$. Note also that $\Xi_0(\delta_0, \delta_1, \ldots, \delta_K) = \chi(x_0)$. For $\ell \geq 1$, compute as follows:

$$\begin{aligned}
\Xi_\ell(\delta) &= \sum_{i=0}^{\ell} (-1)^{\ell-i} \tbinom{\ell}{i} \chi(x_i) \\
&= \sum_{i=0}^{\ell} (-1)^{\ell-i} \tbinom{\ell}{i} \left[\chi(x_0) + \chi'(x_0)(x_i - x_0) + \sum_{k\geq 2} \frac{1}{k!} \chi^{(k)}(x_0)(x_i - x_0)^k \right] \\
&= \chi'(x_0) \sum_{i=0}^{\ell} (-1)^{\ell-i} \tbinom{\ell}{i} x_i + \sum_{i=0}^{\ell} (-1)^{\ell-i} \tbinom{\ell}{i} \sum_{k\geq 2} \frac{1}{k!} \chi^{(k)}(x_0)(x_i - x_0)^k \\
&= \chi'(x_0)\delta_\ell + \sum_{i=0}^{\ell} (-1)^{\ell-i} \tbinom{\ell}{i} \sum_{k\geq 2} \frac{1}{k!} \chi^{(k)}(x_0) \Big[\sum_{j=1}^{i} \tbinom{i}{j} \delta_j \Big]^k \\
&= \chi'(x_0)\delta_\ell + \sum_{k\geq 2} \frac{1}{k!} \sum_{i=0}^{\ell} (-1)^{\ell-i} \tbinom{\ell}{i} \sum_{j_1,\ldots,j_k \geq 1} \tbinom{i}{j_1} \cdots \tbinom{i}{j_k} \chi^{(k)}(x_0)(\delta_{j_1}, \ldots, \delta_{j_k}) \\
&= \chi'(x_0)\delta_\ell + \sum_{k\geq 2} \frac{1}{k!} \sum_{j_1,\ldots,j_k \geq 1} \left[\sum_{i=0}^{\ell} (-1)^{\ell-i} \tbinom{\ell}{i} \tbinom{i}{j_1} \cdots \tbinom{i}{j_k} \right] \chi^{(k)}(x_0)(\delta_{j_1}, \ldots, \delta_{j_k}),
\end{aligned}$$

where we have repeatedly used the multi-linearity of $\chi^{(k)}(x_0)$.

Notice that, for fixed $j_1, \ldots, j_k$, $\binom{i}{j_1} \cdots \binom{i}{j_k}$ is a polynomial in i of degree $j_1 + \cdots + j_k$. Then, by (2.9), $\sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} \binom{i}{j_1} \cdots \binom{i}{j_k}$ is an ℓ-th order difference of uniform samples of a degree $j_1 + \cdots + j_k$ polynomial, which vanishes when $j_1 + \cdots + j_k < \ell$.

Consequently, the ℓ-th component of Ξ has no (linear or nonlinear) terms of weight less than ℓ, thus

$$D^\nu \Xi_\ell|_{(x_0,0,\ldots,0)} = 0, \quad \text{weight}(\nu) < \ell \,.$$

The same conditions hold with Ξ replaced by Ξ^{-1}—simply replace χ with χ^{-1} in the derivation above.

We now know that the ℓ-th component of both Ξ and Ξ^{-1} do not have terms of weight *strictly* less than ℓ, the proximity condition of Ψ says that its ℓ-th component Ψ_ℓ does not have terms of weight ℓ and lower. These facts alone only guarantee that $\overline{\Psi}_\ell$ does not have terms of weight $\ell - 1$ and lower. (The second part of the proof explain this along the way.)

Step 2. To complete the proof we now show that all weight ℓ terms in $\overline{\Psi}_\ell$ vanish. Assume that Ψ satisfies the order k differential proximity condition. By Remark 7 and Step 1, we have for

$\ell = 2, \dots, k$,

$$\begin{aligned}
&\overline{\Psi}_\ell(\overline{\delta}) = \Xi_\ell(\Psi \circ \Xi^{-1}(\overline{\delta})) \\
(4.3) \qquad &= \sum_{\mathrm{weight}(\nu)=\ell} \frac{1}{\nu!} D^\nu \Xi_\ell|_{(x_0,0,\dots,0)} \Big(\Psi_1(\Xi^{-1}(\overline{\delta}))^{\nu_1}, \dots, \Psi_\ell(\Xi^{-1}(\overline{\delta}))^{\nu_\ell}\Big) + (\text{weight} > \ell \text{ terms}).
\end{aligned}$$

For each $i = 1, \dots, \ell$, again by Remark 7 and Step 1,

$$\Psi_i(\Xi^{-1}(\overline{\delta})) = \frac{1}{2^i} \sum_{\mathrm{weight}(\eta)=i} \frac{1}{\eta!} D^\eta \Xi_i^{-1}|_{(\overline{x}_0,0,\dots,0)} \overline{\delta}^\eta + (\text{weight} > i \text{ terms}). \tag{4.4}$$

An inspection then reveals that the only weight ℓ terms in $\overline{\Psi}_\ell(\overline{\delta})$ are

$$\begin{aligned}
&\sum_{\mathrm{weight}(\nu)=\ell} \frac{1}{\nu!} D^\nu \Xi_\ell \left(\Big(\frac{1}{2^1} \sum_{\mathrm{weight}(\eta)=1} \frac{1}{\eta!} D^\eta \Xi_i^{-1} \overline{\delta}^\eta\Big)^{\nu_1}, \dots, \Big(\frac{1}{2^\ell} \sum_{\mathrm{weight}(\eta)=\ell} \frac{1}{\eta!} D^\eta \Xi_i^{-1} \overline{\delta}^\eta\Big)^{\nu_\ell} \right) \\
(4.5) \qquad &= \frac{1}{2^\ell} \sum_{\mathrm{weight}(\nu)=\ell} \frac{1}{\nu!} D^\nu \Xi_\ell \left(\Big(\sum_{\mathrm{weight}(\eta)=1} \frac{1}{\eta!} D^\eta \Xi_i^{-1} \overline{\delta}^\eta\Big)^{\nu_1}, \dots, \Big(\sum_{\mathrm{weight}(\eta)=\ell} \frac{1}{\eta!} D^\eta \Xi_i^{-1} \overline{\delta}^\eta\Big)^{\nu_\ell} \right).
\end{aligned}$$

By yet another inspection, we see that by virtue of the chain rule the weight ℓ terms in the Taylor expansion of $(\Xi \circ \Xi^{-1})_\ell$ are given by the summation after the $2^{-\ell}$ factor in (4.5). But $\Xi \circ \Xi^{-1} =$ identity, so any nonlinear term in its Taylor expansion must vanish. In other words, all the nonlinear (i.e. degree > 1) terms in (4.5) vanish. This implies that the Taylor expansion of $\overline{\Psi}_\ell(\overline{\delta})$ has the linear term $2^{-\ell}\overline{\delta}_\ell$ as its only weight ℓ term, and all other terms, linear or nonlinear, are of weight strictly greater than ℓ. In other words, $\overline{\Psi}$ satisfies the same differential proximity condition as Ψ. □

It is worth stressing the role of the Π_k reproduction property of S_{lin} in the coordinate independence proof above: it induces a kind of "upper-triangular" structure in Ψ_{lin} (Lemma 6) and enters the proof in Step 2 above. In particular, the dyadic eigenvalues in Ψ_{lin} are the key to the derivation of (4.5). Indeed, (4.5) implies that as far as the lowest weight terms (i.e. weight ℓ) in the ℓ-th component are concerned, the map $\Xi \circ \Psi \circ \Xi^{-1}$ is the same as $2^{-\ell}\Xi \circ \Xi^{-1}$.

5. The log-exp scheme on Surfaces

As an application of Theorem 8, we show that the single base-point log-exp scheme introduced in [9] does not satisfy the differential proximity condition. Consequently, thanks to Theorem 8, we can conclude a breakdown of smoothness equivalence in the single base-point scheme.

The paper [3] studies the proximity condition for the single base-point schemes defined by general retraction maps. Since [3] predates the development of Theorem 8, the results therein were derived from the original proximity condition. As the discussion in Section 3 hinted, the breakdown results based on the original proximity condition from [3] easily imply corresponding breakdown results based on our differential proximity condition. Therefore, the anticipated breakdown of smoothness equivalence in the more general setting once again follows from Theorem 8.

As the computations in [3] are rather involved, we present here the special case of the single base-point log-exp scheme based on the C^5, symmetric B-spline, whose subdivision mask is

$$(a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_4) = \frac{1}{64}(1, 7, 21, 35, 35, 21, 7, 1)\,, \tag{5.1}$$

and in the simple case where M is a two-dimensional Riemannian manifold. In this case,

$$q_0(x_0, x_1, x_2, x_3) = \exp_{x_2}\left(a_4 \log_{x_2}(x_0) + a_2 \log_{x_2}(x_1) + a_{-2} \log_{x_2}(x_3)\right), \tag{5.2a}$$

$$q_1(x_0, x_1, x_2, x_3) = \exp_{x_1}\left(a_3 \log_{x_1}(x_0) + a_{-1} \log_{x_1}(x_2) + a_{-3} \log_{x_1}(x_3)\right), \tag{5.2b}$$

$$Q(x_0, x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} q_1(x_0, x_1, x_2, x_3) \\ q_0(x_0, x_1, x_2, x_3) \\ q_1(x_1, x_2, x_3, x_4) \\ q_0(x_1, x_2, x_3, x_4) \\ q_1(x_2, x_3, x_4, x_5) \\ q_0(x_2, x_3, x_4, x_5) \end{pmatrix}, \tag{5.3}$$

and $\Psi = (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5)$ is defined according to (2.7).

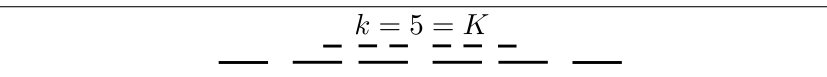


FIGURE 2. Minimal invariant neighborhood of the C^5 B-Spline subdivision scheme

Theorem 11. *The nonlinear scheme S defined by (5.2) satisfies the C^5-equivalence property if and only if the manifold M has vanishing curvature.*

One direction is clear, for suppose M has vanishing curvature, we may then choose local coordinates about any point in M in which the Riemannian metric is the Euclidean metric. But in these coordinates, S coincides with the (linear) C^5 B-spline scheme.

Now assume that M has non-zero curvature at the point x_0. Then by Theorems 8 and 10 it suffices to choose coordinates centered at x_0 in which the derivative

$$D^{\nu}\Psi_5|_{(x_0,0,\ldots,0)}\,, \text{ for } \nu = (1, 2, 0, 0, 0) \tag{5.4}$$

does not vanish.

Notice that, although ν has weight 5, it has degree 3. Consequently, to compute this derivative, we need only compute the Taylor expansion of Ψ_5 up to order 3 and weight 5 in some coordinate system.

The computations are vastly simplified if we perform them in *Riemann normal coordinates* centered at x_0. We merely summarize the results from Riemannian geometry we need. (A detailed treatment of normal coordinates is given in Chapter 4 of [10] as well as in [1], particularly pages 41–42).)

Let $x = (u, v)$ denote normal coordinates on $\mathbb{R}^2$ centered at the origin. Let (u, v, U, V) denote the corresponding coordinates on the tangent bundle TM, where (U, V) are the components of the tangent vector based at (u, v). *Riemann's Theorem* then states that in these coordinates the coefficients of the Riemannian metric are given by

$$\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{K_0}{3} \begin{pmatrix} v^2 & uv \\ uv & u^2 \end{pmatrix}$$

where K_0 denotes the Gauss curvature at $(0, 0)$. A standard computation using this formula and the differential equations for geodesics yields the following Taylor expansion for the exponential map about $(0, 0, 0, 0)$ up to degree 3 in (u, v, U, V):

$$\exp_{(u,v)}(U, V) \approx (u, v) + (U, V) + \frac{2}{3} K_0 \det \begin{pmatrix} U & V \\ u & v \end{pmatrix} \cdot (V, -U) \ .$$

From this, one finds that up to degree 3 in (u_0, v_0, u, v), the Taylor expansion of log is given by

$$\log_{(u_0,v_0)}(u, v) \approx (u, v) - (u_0, v_0) - \frac{2}{3} K_0 \det \begin{pmatrix} u_0 & v_0 \\ u & v \end{pmatrix} \cdot (-(v - v_0), (u - u_0)) \, .$$

Setting $\delta_\ell = (\delta_{\ell,u}, \delta_{\ell,v})$, substituting the expansions for exp and log into the definition of q_σ yields the Taylor expansion of q_σ up to degree 3. Substituting these Taylor expansions into Ψ_5, and dropping all terms in δ_ℓ of degree larger than 3 and weight larger than 5 yields (after a straightforward, but lengthy computation) the formula

$$\Psi_5(\delta) \approx \frac{1}{2^5}(\delta_{5,u}, \delta_{5,v}) + \frac{7}{16} K_0 \left\{ \det \begin{pmatrix} \delta_{1,u} & \delta_{1,v} \\ \delta_{2,u} & \delta_{2,v} \end{pmatrix} (-\delta_{2,v}, \delta_{2,u}) + \det \begin{pmatrix} \delta_{1,u} & \delta_{1,v} \\ \delta_{3,u} & \delta_{3,v} \end{pmatrix} (-\delta_{1,v}, \delta_{1,u}) \right\} .$$

The weight 5 terms are non-zero exactly when $K_0 \neq 0$, so Theorem 11 is proved. This formally disproves the smoothness equivalence conjecture first posted in [9].

While Theorem 11 says that non-vanishing curvature is the root cause of the C^5-breakdown in the nonlinear scheme defined by (5.1)-(5.2), one can show by a similar computation that the same scheme satisfies C^4-equivalence *regardless of the curvature of* M. In [21, 3], such a C^4-equivalence property was found to be attributable to *both* a special property of the exponential map and the dual time-symmetry property of the scheme (5.1)-(5.2). More precisely,

- If one replaces the exponential map by an arbitrary retraction map, then the resulted scheme will satisfy the C^2-equivalence property but suffer a C^3-breakdown on a general manifold.
- If one replaces the underlying linear scheme by a stable C^4 linear subdivision scheme *without* a dual time-symmetry, then the resulted scheme will satisfy a C^3-equivalence property but suffer a C^4-breakdown on a general manifold.

To illustrate the latter point, we next consider the single base-point log-exp scheme based on the C^6 B-spline, whose subdivision mask is

$$(a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_4) = \frac{1}{128}(1, 8, 28, 56, 70, 56, 28, 8, 1) \, . \tag{5.5}$$

In this case,

$$q_0(x_0, x_1, x_2, x_3, x_4) = \exp_{x_2} \left(a_4 \log_{x_2}(x_0) + a_2 \log_{x_2}(x_1) + a_{-2} \log_{x_2}(x_4) + a_{-4} \log_{x_2}(x_5) \right) , \tag{5.6a}$$

$$q_1(x_0, x_1, x_2, x_3) = \exp_{x_1} \left(a_3 \log_{x_1}(x_0) + a_{-1} \log_{x_1}(x_2) + a_{-3} \log_{x_1}(x_3) \right) , \tag{5.6b}$$

$$Q(x_0,x_1,x_2,x_3,x_4,x_5,x_6) = \begin{pmatrix} q_1(x_0,x_1,x_2,x_3) \\ q_0(x_0,x_1,x_2,x_3,x_4) \\ q_1(x_1,x_2,x_3,x_4) \\ q_0(x_1,x_2,x_3,x_4,x_5) \\ q_1(x_2,x_3,x_4,x_5) \\ q_0(x_2,x_3,x_4,x_5,x_5) \\ q_1(x_3,x_5,x_6,x_6) \end{pmatrix}, \tag{5.7}$$

and $\Psi = (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6)$ is defined according to (2.7).

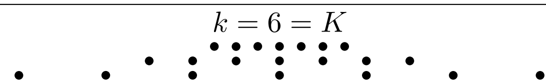


FIGURE 3. Minimal invariant neighborhood of the C^6 B-Spline subdivision scheme

Note that the underlying scheme in this case is even smoother than before (C^6 instead of C^5), and it has a primal symmetry. However, the resulted nonlinear scheme, bsaed on the single base-point strategy, fail to inherit such a primal symmetry.

Theorem 12. *The nonlinear scheme S defined by* (5.6) *satisfies the C^4-equivalence property if and only if the manifold M has vanishing curvature.*

One direction is clear, for suppose M has vanishing curvature, we may then choose local coordinates about any point in M in which the Riemannian metric is the Euclidean metric. But in these coordinates, S coincides with the (linear) C^6 B-spline scheme.

Now assume that M has non-zero curvature at the point x_0. Then by Theorems 8 and 10 it suffices to choose coordinates centered at x_0 in which the derivative

$$D^\nu \Psi_4|_{(x_0,0,\ldots,0)}\,, \text{ for } \nu = (2,1,0,0,0) \tag{5.8}$$

does not vanish.

Notice that, although ν has weight 4, it has degree 3. Consequently, to compute this derivative, we need only compute the Taylor expansion of Ψ_4 up to order 3 and weight 4. We proceed as before, we substitute the Taylor expansions for q_0 and q_1 into Ψ_4, and drop all terms in δ_ℓ of degree larger than 3 and weight larger than 4 to arrive at the expansion

$$\Psi_4(\delta) \approx \frac{1}{2^4}(\delta_{4,u}, \delta_{4,v}) + \frac{1}{3}K_0 \det \begin{pmatrix} \delta_{1,u} & \delta_{1,v} \\ \delta_{2,u} & \delta_{2,v} \end{pmatrix} (-\delta_{1,v}, \delta_{1,u})$$

The weight 4 terms are non-zero exactly when $K_0 \neq 0$, so Theorem 12 is proved.

REFERENCES

[1] R. L. Bishop. Riemannian manifolds. arXiv:1303.5390v2 [math.DG], 2013.

[2] T. Duchamp, G. Xie, and T. P.-Y. Yu. A new proximity condition for manifold-valued subdivision schemes. Manuscript, 2013.

[3] T. Duchamp, G. Xie, and T. P.-Y. Yu. Single basepoint subdivision schemes for manifold-valued data: Time-symmetry without space-symmetry. *Foundations of Computational Mathematics*, pages 1–36, 2013.

[4] P. Grohs. Smoothness equivalence properties of univariate subdivision schemes and their projection analogues. *Numer. Math.*, 113(2):163–180, 2009.
[5] P. Grohs. Smoothness of interpolatory multivariate subdivision in Lie groups. *IMA Journal of Numerical Analysis*, 29(3):760–772, 2009.
[6] P. Grohs. A general proximity analysis of nonlinear subdivision schemes. *SIAM Journal on Mathematical Analysis*, 42(2):729–750, 2010.
[7] P. Grohs. Stability of manifold-valued subdivision schemes and multiscale transformations. *Constructive Approximation*, 32(3):569–596, 2010.
[8] P. Grohs. Finite elements of arbitrary order and quasiinterpolation for Riemannian data. SAM Report 56, ETH Zürich, 2011. submitted.
[9] I. Ur Rahman, I. Drori, V. C. Stodden, D. L. Donoho, and P. Schröder. Multiscale representations for manifold-valued data. *Multiscale Modeling and Simulation*, 4(4):1201–1232, 2005.
[10] M. Spivak. *A comprehensive introduction to differential geometry*, volume 2. Publish or Perish, Inc., 3rd edition, 2005.
[11] J. Wallner. Smoothness analysis of subdivision schemes by proximity. *Constructive Approximation*, 24(3):289–318, 2006.
[12] J. Wallner and N. Dyn. Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Computer Aided Geometric Design*, 22(7):593–622, 2005.
[13] J. Wallner, E. Nava Yazdani, and P. Grohs. Smoothness properties of Lie group subdivision schemes. *Multiscale Modeling and Simulation*, 6(2):493–505, 2007.
[14] J. Wallner, E. Nava Yazdani, and A. Weinmann. Convergence and smoothness analysis of subdivision rules in Riemannian and symmetric spaces. *Advances in Computational Mathematics*, 34(2):201–218, 2011.
[15] G. Xie and T. P.-Y. Yu. Smoothness analysis of nonlinear subdivision schemes of homogeneous and affine invariant type. *Constructive Approximation*, 22(2):219–254, 2005.
[16] G. Xie and T. P.-Y. Yu. Smoothness equivalence properties of manifold-valued data subdivision schemes based on the projection approach. *SIAM Journal on Numerical Analysis*, 45(3):1200–1225, 2007.
[17] G. Xie and T. P.-Y. Yu. Smoothness equivalence properties of general manifold-valued data subdivision schemes. *Multiscale Modeling and Simulation*, 7(3):1073–1100, 2008.
[18] G. Xie and T. P.-Y. Yu. Smoothness equivalence properties of interpolatory Lie group subdivision schemes. *IMA Journal of Numerical Analysis*, 30(3):731–750, 2009.
[19] G. Xie and T. P.-Y. Yu. Approximation order equivalence properties of manifold-valued data subdivision schemes. *IMA Journal of Numerical Analysis*, 2011.
[20] G. Xie and T. P.-Y. Yu. Invariance property of the proximity condition in nonlinear subdivision. *Journal of Approximation Theory*, 164(8):1097–1110, 2012.
[21] E. Nava Yazdani and T. P.-Y. Yu. On Donoho's Log-Exp subdivision scheme: Choice of retraction and time-symmetry. *Multiscale Modeling and Simulation*, 9(4):1801–1828, 2011.

Tom Duchamp, Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195-4350, U.S.A.

E-mail address: duchamp@math.washington.edu

Gang Xie, Department of Mathematics, East China University of Science and Technology, Shanghai, China, 200237

E-mail address: rpi1004@ecust.edu.cn

Thomas Yu, Department of Mathematics, Drexel University, 3141 Chestnut Street, 206 Korman Center, Philadelphia, PA 19104, U.S.A.

E-mail address: yut@drexel.edu