Numerical Methods for Biomembranes based on PL Surfaces

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Abstract. The shapes of phospholipid bilayer biomembranes are modeled by the celebrated Canham-Evans-Helfrich model as *constrained Willmore minimizers*. Several numerical treatments of the model have been proposed in the literature, one of which was used extensively by biophysicists over two decades ago to study real lipid bilayer membranes. While the key ingredients of this algorithm are implemented in Brakke's well-known surface evolver software, to the best of our knowledge the glory details of the algorithm was never explained by either the geometers who invented it or the biophysicists who used it. As such, most of the computational results claimed in the biophysics literature are difficult to reproduce. In this note, we give an exposition of this not widely known method, connect it with some related ideas in the literature, and propose a modification of the original method based on replacing mesh smoothing with harmonic energy regularization. We present a theoretical finding and related computational observations explaining why such a smoothing or regularization step is indispensable for the success of the algorithm. A software package called Wmincon is available for reproducing the experiments in this and related articles.

1 Introduction

Lipid bilayer is arguably the most elementary and indispensable structural component of biological membranes which form the boundary of all cells. It is known since the seminal work of Canham [5], Helfrich [12] and Evans [9] in the 70s that bending elasticity, induced by curvature, plays the key role in driving the geometric configurations of such membranes.

The so-called spontaneous curvature model of Helfrich (referred simply to as the **Helfrich problem**) suggests that a biomembrane surface S configures itself so as to minimize $\int_S H^2 dA$ subject to the area, volume and area difference (related to the bilayer characteristics) constraints, i.e. S solves the variational problem

$$\min_S W[S] := \int_S H^2\, dA \text{ s.t. } \begin{cases} \text{(i)} \quad A[S] := \int_S 1\, dA = A_0, \\ \text{(ii)} \quad V[S] := \frac{1}{3}\int_S [x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dA = V_0, \\ \text{(iii)} \quad M[S] := -\int_S H\, dA = M_0. \end{cases}$$

Here $H = (\kappa_1 + \kappa_2)/2$ is the mean curvature. In (ii), $V[S]$ is the enclosing volume as a surface integral of S. The connection of (iii) to bilayer area difference comes from the relation $-\int_S H dA = \lim_{\varepsilon\to 0} \frac{1}{4\varepsilon}(\text{area}(S_{+\varepsilon}) - \text{area}(S_{-\varepsilon}))$,

where $S_{+\varepsilon}$ and $S_{-\varepsilon}$ are the 'ε-offset surfaces',[1] and that the thickness of the lipid bilayer, 2ε, is negligible compared to the size of the vesicle. The constraint values A_0, V_0 and M_0 are determined by physical conditions (e.g. temperature, concentration). $W[S]$ is called the **Willmore energy** of the surface S. When the area-difference constraint (iii) is omitted, the model is referred to as the **Canham problem**. When even the volume constraint (ii) is omitted, there is essential no constraint as W is scale-invariant; in this case the area constraint (i) only fixes the scale, and we refer to the variational problem as the **Willmore problem**. The mathematical depth and physical relevance of these problems make them timeless challenges for geometric analysts and computational mathematicians.

It is observed experimentally that no topological change occurs in any accessible time-scale, so we aim to solve any of the Helfrich, Canham or Willmore problems when S is assumed to be an orientable closed surface with a fixed genus g. Spherical, i.e. $g = 0$, vesicles are the most ubiquitous among naturally occurring biomembranes, although higher genus ones exist also [18,15,19]. The Helfrich and Canham models explain the large variety of shapes observed in even a closed vesicle with a spherical topology [16,19].

Several numerical treatments of these models have been proposed in the literature, one of which was used extensively by biophysicists over two decades ago to study real lipid bilayer membranes. While the key ingredients of this algorithm are implemented in Brakke's well-known surface evolver software [4], the overall algorithm seems to be never explained clearly by either the geometers who invented it [13,4,11] or the biophysicists who used it [18,19]. As such, most of the computational results claimed in the biophysics literature are difficult to reproduce.

In this note, we give an exposition on the numerical method, explain its connections to ideas developed in the applied geometry literature [17] and ideas developed in the FEM literature [3], present some theoretical and empirical findings, propose a modification of the original method based on the idea of harmonic energy regularization, and discuss open issues.

2 Direct Minimization of Discrete Willmore Energy

A standard approach to represent surfaces of arbitrary topology is piecewise linear (PL) surfaces. A PL surface can be specified by a mesh $M = (\mathcal{V}, \mathcal{F})$ where $\mathcal{V} \in \mathbb{R}^{\#V \times 3}$ records the 3-D coordinates of the vertices of the control mesh, $\#V$ denotes the total number of vertices, and $\mathcal{F} \in \mathcal{I}^{\#F \times 3}$ is a list of triplets of indices from $\mathcal{I} := \{1, \ldots, \#V\}$ which records the bounding vertices of each of the $\#F$ triangle faces in the mesh M. We assume that the PL surfaces realized by the mesh is closed and orientable.

[1] We assume that the normal of any closed orientable surface points outward, so that $H < 0$ for a sphere.

In a numerical method, our view is usually that $\mathcal{F}$ is fixed and $\mathcal{V}$ varies. This fits the framework of our variational problems well, as fixing $\mathcal{F}$ also fixes the genus of the surface, and varying $\mathcal{V}$ means we find the embedding of $\mathcal{F}$ – viewed as an abstract simplicial complex – that optimizes the Willmore energy under the corresponding constraint(s).

A PL surface has a well-defined area A and enclosing volume V, but no well-defined normals or mean curvatures, hence it also does not have a well-defined total mean curvature M or Willmore energy W. As such, any numerical method for the Willmore, Canham, or Helfrich problems based on PL surfaces may be classified as a "nonconforming finite element method".

One way to derive such a method is to find a definition of M and W for PL surfaces that is consistent in the sense that for any 'reasonable' sequence of subsequently finer triangulations of a smooth surface, the PL W or M-energies would converge to the corresponding continuous energies of the smooth surface. We then solve any version of the variational problems numerically by fixing a face list $\mathcal{F}$ with the presumed topology and desired resolution, then proceed to solve the corresponding finite-dimensional constrained optimization problem with $\mathcal{V}$ as variables. This 'optimizing a discretization' approach is used in [13] for solving the Willmore problem. It is mentioned in the aforementioned biophysics work that the approach is extended to solve the Canham and Helfrich problems, however almost no computational details are found in these physics papers. Some of the details can be found, however, in the manual of the Surface Evolver. Another interesting application of this approach is found in [11] for illustrating a least bending sphere eversion.

There are at least half a dozen discrete mean curvatures and discrete Willmore energies proposed in the literature. One of them is the one we first learned from the paper by Hsu-Kusner-Sullivan [13], denoted here by H_{HKS} and W_{HKS}, based on a particular way of discretizing the area variation characterization of the mean curvature. Recall that for any smooth orientable surface S with continuous unit normals denoted by $\mathbf{n}(x)$, $x \in S$, we have $\frac{d}{dt}\big|_{t=0}\text{Area}(S_t) = -2\int_S h(x)H(x)dA$, where $S_t := \{x + th(x)\mathbf{n}(x) : x \in S\}$ and h is any scalar field on S. To find a consistent definition of mean curvature for PL surfaces, we need to also come up with a definition of normals for PL surfaces. The idea is to define 'discrete normals' $\mathbf{n}(v)$, 'discrete mean curvatures' $H(v)$, and local areas $a(v)$ at the vertices of a PL surface so that $\sum_v a(v) = A =$ area of the PL surface, and $\nabla_v A \cdot \mathbf{n}(v) = -2H(v)a(v)$, $\forall v$. Once $a(v)$ is assigned, then $\mathbf{n}(v)$ and $H(v)$ can be chosen so that $\overrightarrow{H}(v) := H(v)\mathbf{n}(v) = -\frac{\nabla_v A}{2a(v)}$. The local area in H_{HKS} is $a(v) = \text{Area}(\text{star}(v))/3$. The discrete mean curvature introduced in [17], denoted here by $H_{\text{MDSB}}(v)$, corresponds to $a(v) = \text{Area}(\text{Voronoi cell around } v)$.[2] In either case, the discrete

[2] The paper [17] derived the same formula based on the characterization of mean curvature by the Laplace-Beltrami operator: $\Delta_S X(x) = \overrightarrow{H}(x)$, $x \in S$, where $X : S \to \mathbb{R}^3$ is the position function of the surface S.

Willmore energy $W_{\text{HKS/MDSB}}$ and total mean curvature $M_{\text{HKS/MDSB}}$ are defined as $W = \sum_v H(v)^2 a(v)$ and $M = \sum_v H(v) a(v)$, respectively.

Yet another discrete Willmore energy is proposed by Bobenko [1], $W_{\text{Bobenko}} = \sum_{e \in \text{edge}} \beta(e) - \pi \# V$, where $\beta(e)$ is the angle formed by the two circumscribed circles of the two triangles sharing the edge e. Interestingly, the original paper [1] puts a heavy emphasis on its invariance property under Möbius transformation, but leaves almost no trace of why this energy has anything to do with the continuous Willmore energy. However, it is not hard to check that the energy is consistent with the continuous W-energy in a certain sense [2].

How good are these PL methods when used for solving any of the variation problems? One may think that any of these PL methods is simply less accurate than the higher order methods based on subdivision surfaces (SS) [10,6,7]. Here we present a computational experiment comparing three methods for solving the $g = 1$ Willmore problem, based on a direct minimization of (i) W_{HKS}, (ii) W_{Bobenko}, and (iii) the (true) Willmore energy of Loop SS. In each case, a regularly triangulated (i.e. all vertices have valence 6) torus with 5 and 9 grid points in the minor and major circle directions is used. By the Marques-Neves theorem, the minimum Willmore energy is $2\pi^2$. The approximate minimizers based on our WMINCON package is shown below:

(a) initial mesh | (b) $W_{\text{HKS}} = 8.85 \ll 2\pi^2$ | (c) $W_{\text{Bobenko}} = 12.57 \approx 4\pi \ll 2\pi^2$ | (d) control mesh of W-minimizing Loop SS | (e) $W = 19.83 \gtrapprox 2\pi^2$

Fig. 1. Failure of W_{HKS} and W_{Bobenko} for solving the genus 1 Willmore problem: method converges to a PL surface unrelated to a Clifford torus, and with an energy way below $2\pi^2$. One can change 'HKS' to 'MDSB' or 'EffAreaCur' in our code to observe a similar failure. For SS methods, one can change 'Loop' to 'C2g0' [6] to observe a even higher accuracy.

In Figure 1(e), we minimize the true Willmore energy, accurately computed by a higher order numerical quadrature, of a finite dimensional family of Loop SS. In this case, the minimum cannot be smaller than $2\pi^2$. W_{HKS} and W_{Bobenko}, however, would presumably be a good approximation to the true Willmore energy *only when applied to a PL surface that is a reasonable discretization of a smooth surface*. The above experiment exposes an unfortunate phenomenon: there are PL surfaces which are by no means a 'reasonable discretization' of any smooth torus but yet have a $W_{\text{HKS/Bobenko}}$-energy way less than $2\pi^2$.

We note that the computational result in [2, Figure 8] seems to suggest that a minimizer of W_{Bobenko} would be an accurate approximation of a Clifford torus, which is the exact opposite of what we report here. This apparent

contradiction may be caused by the non-smoothness of W_{Bobenko}; indeed we could not get the 'spherical torus' in Figure 1(c) without the help of a non-smooth optimization solver (we use GRANSO [8]). See also Proposition 1 below.

Addressing the failure. A fundamental shortcoming of H_{HKS} is already addressed in [13] and the same behavior is exhibited by H_{MDSB}: if v is a vertex over an n-gon (assumed regular for simplicity), then $H_{\text{HKS}}(v)$ first increases as the height of v over the n-gon increases, but then decreases after a certain height. In other words, after a certain threshold $H_{\text{HKS/MDSB}}(v)$ decreases despite the PL surface at v becoming sharper and sharper – a behavior that is particularly dangerous for variational problems. See Figure 2. Two alternatives, under the name of 'effective area curvature' and 'normal curvature' in Brakke's surface evolver manual, were introduced to alleviate this drawback: $\overrightarrow{H}_{\text{EffAreaCur}} = -\frac{\nabla_v A}{2\|\nabla_v V\|}$, $\overrightarrow{H}_{\text{NormalCur}} = -\frac{\nabla_v A \|\nabla_v A\|}{2\langle \nabla_v V, \nabla_v A\rangle}$, where V is the enclosing volume of the PL surface.

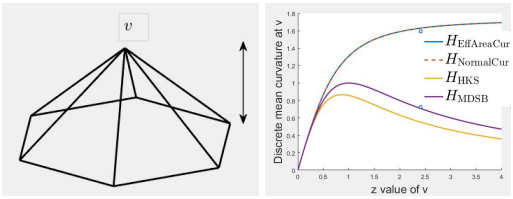
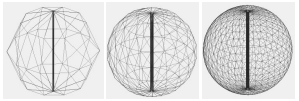


Fig. 2. Pointwise behavior of four discrete mean curvatures

Note that this is a pointwise issue; we shall be back to the global Willmore problem after mentioning yet another not so well-known pointwise issue: all the discrete mean curvatures introduced above, except H_{MDSB}, converge to the correct value $(\kappa_1+\kappa_2)/2$ when applied to samples of the graph of $(u, v) \mapsto \frac{1}{2}(\kappa_1 u^2+\kappa_2 v^2)$ at a small regular n-gon near $(0, 0)$ *only when* $n = 6$. However, computational experiences suggest that this problem is forgivable as long as we use triangulations dominated by valence 6 vertices; this can be guaranteed by repeatedly applying midpoint subdivision to a coarse triangulation.

In [7], we prove the following:

Proposition 1. For every grid size (m, n), there is a regularly triangulated family of 'almost spherical tori' $T_{m,n,\varepsilon}$, each with a hole of diameter $\sim \varepsilon$, such that as $\varepsilon \downarrow 0$, $W_{\text{Bobenko}}(T_{m,n,\varepsilon})$ decreases monotonically to $4\pi \ll 2\pi^2 = W(\text{Clifford torus})$.


(5, 10, .01) (10, 20, .05) (20, 40, .1)

This negative result shows that increasing the grid sizes m and n *cannot* resurrect the failure seen in Figure 1(c), despite the fact that W_{Bobenko} is a consistent discretization of the Willmore energy [2]. The situation is reminiscent of the "consistency without stability does not imply convergence" situation in numerical PDEs.

We conjecture that similar negative results can be established for W_{HKS}, W_{MDSB} and $W_{\text{EffAreaCur}}$, with the 'almost spherical tori' in Proposition 1

replaced by some carefully constructed 'almost flat tori', as suggested by our computations (Figure 1(b) and 3(a)) and preliminary calculations.

3 Harmonic Energy Regularization

All failures we observed theoretically or experimentally have one thing in common: triangles with bad aspect ratios develop. This is of course a familiar issue in FEM and mesh generation, but we believe that in the moving surface context here the issue is fundamentally different. In previous works procedures going under the name of 'vertex averaging', 'edge notching' and 'equiangulation', implemented in the surface evolver, were introduced to somehow interfere an optimization process by fixing up 'bad triangles'. Such a numerical method was reported to work for solving the Willmore, Canham and Helfrich problems. While these mesh smoothing procedures are probably well studied in the mesh generation community, we do not know exactly how they should be used to give a successful numerical method. Little implementation details are available in the relevant papers, and such a method seems difficult to analyze mathematically.

We propose a method with a similar spirit but a cleaner formulation, based on adding to any PL W-energy W_{PL} a *harmonic energy* penalization term, i.e. to solve $\min W_{\mathrm{PL}} + \lambda\,\mathcal{E}$ subject to the relevant constraint(s). Here $\mathcal{E} = \int_M \|d\varphi\|^2 dA$ is defined w.r.t. a conformal structure [14, Ch. 8] imposed on the underlying simplicial complex where *all triangles are equilateral*. Unlike the Willmore energy, a PL surface has enough regularity for $\mathcal{E}$ to be well-defined; here we view a PL surface as an immersion $\varphi : M \to \mathbb{R}^3$ of the underlying simplicial complex, now thought of as an abstract Riemann surface M, into $\mathbb{R}^3$. It is not hard to check that $\mathcal{E} = 1/\sqrt{3}\sum_{e\in\mathrm{edge}} \mathrm{length}(e)^2$.

See Figure 3 for the effect of such a regularization on $W_{\mathrm{EffAreaCur}}$ and $W_{\mathrm{NormalCur}}$: it does not just fix up bad triangles, it completely changes the landscape of the optimization problem. The regularized minimizers are much closer to a Clifford torus.

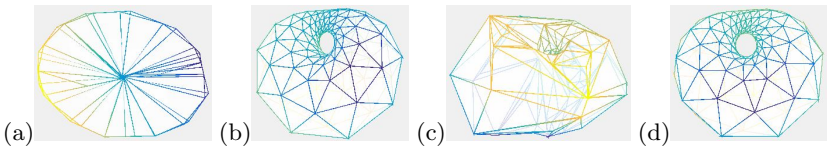


Fig. 3. Genus 1 $W_{\mathrm{EffAreaCur/NormalCur}}$-minimizer with a $(10, 16)$ regular triangulation, with ((b) & (d)) and without ((a) & (c)) $\mathcal{E}$-regularization

Finally, Figure 4 shows how the regularized method performs on the genus 0 Canham problem with reduced volume/isoperimetric ratio $v_0 := \frac{3V_0}{4\pi}(\frac{A_0}{4\pi})^{-\frac{3}{2}}$ in three different intervals known to give the shapes of a stomatocyte, discocyte (red-blood cell) and prolate. The computational result is consistent with what is reported in the biophysics literature, and those produced by

SS methods. Again, harmonic energy regularization is indispensable for the success.

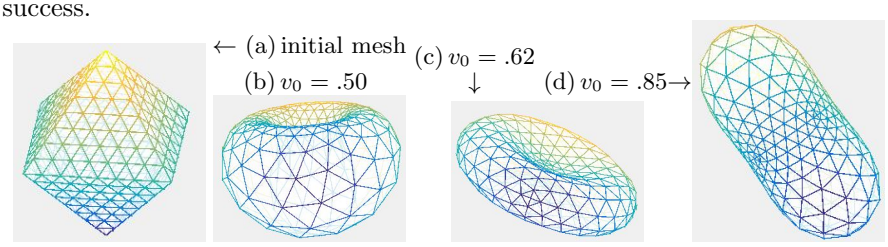


Fig. 4. Genus 0 v_0-constrained $W_{\text{NormalCur}}$-minimizers

4 Conclusion and Open Issues

Our empirical observation – supported by Proposition 1 and various experiments – is that a space of PL surfaces, in contrast to SS, would be lured by any known discrete W-energies to waste its degrees of freedom on long and skinny triangles. Harmonic energy regularization comes to rescue and helps putting the degrees of freedom to good use, namely to optimize the accuracy of approximation to the true solution of the continuous problem.

Open questions pertaining to these observations include: How to choose the penalization parameter λ? How to design a method with a provable Γ-convergence property? Is there a 'perfect' discrete W-energy that does not require any regularization? We believe that it is much harder to establish Γ-convergence results for PL methods than for SS methods. More details can be found in the journal version of this paper [7].

The PL and SS methods discussed here are in the spirit of "minimizing a discretization", whereas the method proposed by Bonito-Nochetto-Pauletti [3] is in the spirit of "discretizing a minimization"; it involves a tricky time- and space-discretization of a Willmore flow that bypass the lack of regularity in PL or piecewise polynomial surfaces. While we are learning about even newer methods for these fundamental geometric problems, possibly based on entirely different ways to represent surfaces, it would be interesting to understand the existing methods more deeply.

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