A Survey of Subdivision Algorithms of Manifold-Valued Data

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Manifold-Valued Data

- Mo-Cap: $[0, T] \rightarrow \text{SO}(3) \times \text{SO}(3) \times \text{SO}(3) \times \text{SO}(2) \times \text{SO}(2) \times \text{SO}(3) \times \text{SO}(3) \times \text{SO}(3) \times \text{SO}(2) \times \text{SO}(2) \times \text{SO}(3) \times \text{SO}(3) \times \text{SO}(3) \times \text{SO}(3)$

  - neck
  - shoulders
  - elbows
  - wrists
  - hips
  - knees
  - ankles
  - spine

- DTI: $[0, L]^3 \rightarrow \text{SPD}(3)$
Linear Subdivision of Real Data

\( (S_{\text{lin}} x)_{2h+\sigma} = \sum_{\ell} a_{2\ell+\sigma} x_{h-\ell}, \quad \sigma = 0, 1, \quad h \in \mathbb{Z} \)
### Symmetry in 1-D linear subdivision

#### ‘Time-Symmetry’:

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>e.g. Dubuc scheme, odd deg. B-Spline</td>
<td>Donoho’s Al scheme, even deg. B-Spline</td>
</tr>
<tr>
<td>( S_{\text{lin}}(x_{-k}) = (S_{\text{lin}}x)_{-k} )</td>
<td>( S_{\text{lin}}(x_{1-k}) = (S_{\text{lin}}x)_{1-k} )</td>
</tr>
<tr>
<td>( a_{-k} = a_k )</td>
<td>( a_{1-k} = a_k )</td>
</tr>
<tr>
<td>Data associated with dyadic points</td>
<td>Data associated with dyadic intervals</td>
</tr>
</tbody>
</table>

#### ‘Space-Symmetry’:

\[
S_{\text{lin}}(ax + b) = aS_{\text{lin}}(x) + b.
\]
Subdivision of SO(3) Data
Subdivision of Manifold-valued Data

2.2.2 Subdivision in Riemannian manifolds. The word Riemannian manifold means a manifold (which for the purposes of Computer Graphics is realized as a piece of $\mathbb{R}^{d}$ or a parametrized surface or a triangulation), where the way of computing the scalar product of two tangent vectors attached to the same point is not the one of ambient space, but is allowed to change smoothly with the point quite independent of the way the manifold lies in some other space. Of course we might as well use the scalar product of ambient space, so every surface in the ordinary sense is a Riemannian manifold. It turns out that the theory dealing with smoothness of limits extends easily to Riemannian manifolds, so for the sake of completeness we give an example of subdivision in terms of geodesics of a Riemannian manifold.

The Poincaré disk model of hyperbolic geometry (here thought to be contained in the unit disk of $\mathbb{R}^2$, see Fig. 5) is a fascinating geometry which appears in Computer Graphics e.g. because it is suitable for visualization of surface-like data which will not fit well into the Euclidean plane. Geodesics are easily computable [Alekseevskij et al. 1993] – they appear as circles which intersect the boundary of the hyperbolic disk orthogonally. The geodesic midpoint $g$-av$_{1/2}(a, b)$ of points $a, b$ is computed by $g$-av$_{1/2}(a, b) = \psi(\phi(a) + \phi(b))$, where $|||x|||^2 = x_1^2 - x_2^2 - x_3^2$, (6)

$\phi(x) = \frac{1}{1-x_2^2} \left( x_2, 2x_1, 2x_2 \right)$, $\psi(x) = \frac{1}{1+x_1^2} \left( x_2, x_3 \right)$. Equ. (6) is sufficient for defining hyperbolic B-spline subdivision analogous to Equ. (2). because $g$-av$_{1/4}(a, b) = g$-av$_{1/2}(a, g$-av$_{1/2}(a, b))$, $g$-av$_{3/4}(a, b) = g$-av$_{1/4}(b, a)$, (7)

The curves which appear in Fig. 5 are the result of hyperbolic B-spline subdivision analogous to $B_3$.

2.2.3 Intrinsic shape properties of geodesic schemes. If a planar subdivision rule $S$ works by corner cutting (the B-spline schemes do), then it enjoys the variation diminishing property: A straight line intersects the edges of each polygon $S_i p$ at least as often as it intersects the edges of the polygon $S_j p$, if $j > i$ ($j = \infty$ and $i = 0$ are allowed). This follows directly from the fact that two line segments have at most one intersection point. Therefore geodesic subdivision rules in a 2-dimensional surface or manifold have the same property, provided geodesic segments intersect in one point only. This is not the case in general, but can be guaranteed for any compact surface if the shorter of the two segments in question is not longer than a certain constant, which depends on the intrinsic curvature of the surface [do Carmo 1992]. This geodesic variation diminishing property is illustrated by Fig. 3.
Subdivision Schemes for Manifold-Valued Data

based on a linear scheme $S_{\text{lin}}$ with mask $(a_{2\ell+\sigma})$

- **Extrinsic**: based on embedding and projection
- **Intrinsic**:
  - Single base-point scheme
    \[
    (Sx)_{2h+\sigma} = \text{Exp}_{x_h} \left( \sum_{\ell} a_{2\ell+\sigma} \text{Log}_{x_h} (x_{h-\ell}) \right), \quad \sigma = 0, 1, \quad h \in \mathbb{Z}
    \]
    Related to a *non-redundant* wavelet-like transform for manifold-valued data due to Donoho et al
  - Two base-point scheme (with $b_{h \pm 1/2}$ judiciously chosen from data $(x_h)$)
    \[
    (Sx)_{2h} = \text{Exp}_{b_{h-1/2}} \left( \sum_{\ell} a_{2\ell+\sigma} \text{Log}_{b_{h-1/2}} (x_{h-\ell}) \right)
    \]
    \[
    (Sx)_{2h+1} = \text{Exp}_{b_{h+1/2}} \left( \sum_{\ell} a_{2\ell+\sigma} \text{Log}_{b_{h+1/2}} (x_{h-\ell}) \right)
    \]
- **Others...**

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Log and Exp maps

Figure 3.5: A manifold, its tangent plane, and the correspondence between a line in the tangent plane and a geodesic in the manifold.

The procedure of fixing a vector in $\theta \in T_{p_0}(M)$ as an initial velocity for a (constant-speed) geodesic establishes an association between $T_{p_0}(M)$ and a neighborhood of $p_0$ in $M$. This association is one-one over a ball of sufficiently small size in $T_{p_0}(M)$ – up to the so-called injectivity radius $\rho$. The association is formally captured by the exponential map $p_1 = \text{Exp}_{p_0}(\theta)$. Within an appropriate neighborhood $N_{p_0}$ of $p_0$, the inverse map – the so-called ‘Logarithm map’ – is well-defined, taking $N_{p_0} \subset M$ into $T_{p_0}(M)$. Formally, this correspondence is written as $\theta = \text{Log}_{p_0}(p_1)$.

We are only interested in manifolds for which Log/Exp maps can be explicitly given; examples will be provided below.

3.2 $M$-valued Interpolatory Approach

Clearly, the interpolatory pyramid $\beta_{j,k} = p(t_{j,k})$ makes just as much sense as in the $\mathbb{R}$-valued case, has the same ‘hard’ redundancies $\beta_{j+1,2k} = \beta_{j,k}$, and the same ‘expected’ redundancies $\beta_{j+1,2k+1} \approx \beta_{j,k}$ for smooth functions. We first discuss how to ‘predict’ coarse-to-fine on manifolds, giving $M$-valued two-scale refinement schemes, and then describe a wavelet pyramid $(\alpha_{j,k})_{j,k}$ removing the redundancy from $(\beta_{j,k})_{j,k}$.

3.2.1 Interpolatory Refinement on Manifolds

Given a sequence $p(k), k \in \mathbb{Z}$ taking values $p(k) \in M$, we can (often) impute data at the half-integers by a scheme which might be called “Deslauriers-Dubuc in the tangent space”. Fix an odd integer $D$, for example 3. To get an imputation $\tilde{p}(1/2)$, we use the data $p(\ell)$ at the $D+1$ integer sites $\ell$ nearest to $1/2$. Letting $p_0 = p(0)$, we then map these points to the tangent plane $T_{p_0}(M)$ via $\theta(\ell) = \text{Log}_{p_0}(p(\ell))$, $\ell = -(D-1)/2, \ldots, (D+1)/2$.

The resulting $\theta(\ell)$ belong to a vector space and it makes sense to add, scale, subtract, and so on. We take a basis $(e_j)$ for this vector space, getting a $d$-dimensional coordinate representation...
Analysis Questions

- ‘Time Symmetry’: $S_{\text{lin}}$ has a primal or dual symmetry $\Rightarrow$ so is $S$
- ‘Space Symmetry’: For symmetric space $M$ with group action by $G$,
  \[
  S \circ g = g \circ S, \quad \forall g \in G
  \]

- Smoothness Equivalence: $S_{\text{lin}}$ is $C^k$ $\Rightarrow$ so is $S$
- Approx. Order Equivalence: $S_{\text{lin}}$ has $O(h^R)$ accuracy $\Rightarrow$ so is $S$
Easy Observations

- All the schemes we consider satisfy ‘space symmetry’ when applied in a Lie group or symmetric space setting.
- If $S_{\text{lin}}$ has a dual symmetry (e.g. even degree B-Spline), then so does the associated single base-point scheme.
- The two base-point scheme $S$, with appropriately chosen base points $b_{h\pm 1/2}$, can preserve both primal and dual symmetry.
Proximity Condition $\Rightarrow$ Smoothness Equivalence

**Definition**

$S$ and $S_{\text{lin}}$ satisfy the order $k$ proximity condition if

$$\|\Delta^{j-1}(Sx - S_{\text{lin}}x)\|_{\infty} \lesssim \Omega_j(x), \quad j = 1, \ldots, k,$$

where $\Omega_j(x) := \sum (\gamma_1, \ldots, \gamma_j) \prod_{i=1}^j \|\Delta^i x\|_{\gamma_i}^{\gamma_i}, \sum_{i=1}^j i \gamma_i = j + 1$.

<table>
<thead>
<tr>
<th>Order $k$</th>
<th>Condition: Order $k - 1$ proximity conditions +</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,1</td>
<td>$|Sx - S_{\text{lin}}x| \lesssim |\Delta x|^2$</td>
</tr>
<tr>
<td>2</td>
<td>$|\Delta^1 Sx - \Delta^1 S_{\text{lin}}x| \lesssim |\Delta x|^3 + |\Delta x||\Delta^2 x|$</td>
</tr>
<tr>
<td>3</td>
<td>$|\Delta^2 Sx - \Delta^2 S_{\text{lin}}x| \lesssim |\Delta x|^4 + |\Delta x||\Delta^3 x| + |\Delta^2 x|^2$</td>
</tr>
<tr>
<td>4</td>
<td>$|\Delta^3 Sx - \Delta^3 S_{\text{lin}}x| \lesssim |\Delta x|^5 + |\Delta x||\Delta^4 x| + |\Delta x|^2|\Delta^3 x| + |\Delta^2 x||\Delta^3 x| + |\Delta x||\Delta^2 x|^2$</td>
</tr>
</tbody>
</table>

**Theorem (Wallner, Dyn, streamlined by Xie and Y.)**

Assume that $S_{\text{lin}}$ is stable and $C^k$, $k \geq 1$. If $S$ and $S_{\text{lin}}$ satisfy the order $k$ proximity condition, then $S$ is also $C^k$.  

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Subdivision of Manifold-valued Data
A stronger proximity condition

**Definition**

S and $S_{lin}$ satisfy the strong order $k$ proximity condition if

$$\|\Delta^{k+1}(Sx - S_{lin}x)\|_\infty \lesssim \Omega_k(x).$$

Obvious: Strong proximity condition $\Rightarrow$ proximity condition.

Proof: $\|\Delta^k x\|_\infty \leq 2^k \|x\|_\infty$

**Theorem**

*The two conditions are equivalent if S and $S_{lin}$ are interpolatory.*

Proof: Use Hermite-Biehler theorem to show $\Delta^k : \ell^\infty \rightarrow \ell^\infty$ is invertible when restricted to the subspace

$$\{(\cdots, 0, d_{-1}, 0, d_0, 0, d_1, 0, d_2, \cdots, )\}.$$
Calculation of Proximity Condition

- Pick a chart $\phi : V \subset M \rightarrow \mathbb{R}^n$, or an embedding $\Phi : M \rightarrow \mathbb{R}^N$
- Former case: $S_{\text{lin}}$ and $S$ act on $\mathbb{R}^n$-valued data
- Latter case: $S_{\text{lin}}$ and $S$ act on $\mathbb{R}^N$-valued data
- Convert subdivision data $x_0, x_1, x_2, x_3, \ldots$ to $D_0, D_1, D_2, \ldots$

\[
D_0 = x_0, \quad D_1 = x_1 - x_0, \quad D_2 = x_2 - 2x_1 + x_0, \ldots
\]

\[
x_k = \sum \binom{k}{j} D_j
\]

- Take Taylor expansion of $S$, based on the Taylor expansions of $\text{Exp}$ and $\text{Log}$ at $x_0$
- Analyze terms (the ‘bad guys’) that cannot be automatically absorbed into $\Omega_k(x)$
Which way (single base-point scheme, two base-point scheme, etc.) of adapting $S_{\text{lin}}$ to manifold-valued data produces a (nonlinear) scheme $S$ which satisfies the order $k$ proximity condition?

Donoho’s conjecture: His (single base-point) scheme is always as smooth as the underlying $S_{\text{lin}}$. 
Result I: interpolatory case

**Theorem (Xie and Y.)**

If $S_{\text{lin}}$ and $S$ are interpolatory, then all the mentioned schemes satisfy order $k$ proximity condition whenever $S_{\text{lin}}$ reproduces $\Pi_k$. Therefore, smoothness and approximation order equivalence hold in the interpolatory case.

Non-interpolatory case much more subtle.
Result II: non-interpolatory + two base-point scheme

Theorem (Xie and Y. 2008)

For any $k$, there exists an efficient way to choose $b_{n+1/2}$ in the two base-point scheme so that $S$ satisfies the proximity condition of order $k$ with the underlying linear scheme $S_{\text{lin}}$.

- Result unreasonably general: no geometric concept (symmetry, curvature, property of the $\text{Exp}$ and $\text{Log}$ maps, whatsoever) enters the proof of the above result!
- In particular, one is free to replace $\text{Exp}_x$ and $\text{Log}_x$ by any $f : TM \to M$
Result III: non-interpolatory + single base-point

Single base-point scheme is another story...

\[(Sx)_{2h+\sigma} = f_{x_h}\left(\sum_{\ell} a_{2\ell+\sigma} g_{x_h}(x_{h-\ell})\right), \quad \sigma = 0, 1\]

**Theorem (Easy)**

* S and S_{lin} satisfy the order 2 proximity condition for any \( f : TM \rightarrow M. \)

**Theorem (Navayazdani, Duchamp and Y.)**

* S and S_{lin} satisfy the order 3 proximity condition if \( f = \text{Exp}. \)

- We don’t really need \( \text{Exp}, \) just that we cannot use any \( f. \)
- E.g. On \( SO(n), \) \( e^z = \frac{1 + \frac{1}{2}z + \frac{1}{12}z^2}{1 - \frac{1}{2}z + \frac{1}{12}z^2} + O(z^5) \) works,
  \[e^z = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z} + O(z^3) \] does not.
**Result III: finer version**

**Theorem (Navayazdani)**

\[ S \text{ and } S_{\text{lin}} \text{ satisfy the order 3 proximity condition if } P_f \equiv 0. \]

**Theorem (Coordinate independence of } P_f = 0 \text{ cond., Y.)**

\[ P_f \text{ is a (type } (1, 3)\text{) tensor associated with } f : TM \to M. \text{ (Note: same type as the curvature tensor } R_f.\text{)} \]

**Theorem (Geometric interpretation of } P_f = 0 \text{ cond., Duchamp)**

\[ P_f = 0 \text{ if and only if } f \text{ and } \exp_f \text{ agree up to 3rd order along } M \subset TM. \]

Recap: no such differential geometric condition on } f \text{ is required in the analysis of the two base-point schemes.}
Having gotten his result, it is natural for Tom Duchamp to speculate that $S$ and $S_{\text{lin}}$ satisfy the order $k$ proximity condition if $f$ and $\exp_f$ agree up to the $k$th order along $M \subset TM$.

This conjecture is far from the truth!
Result IV: non-interpolatory + single base-point

Assume $S$ is the single base-point scheme based on $S_{\text{lin}}$ and a retraction map $f$ with $P_f = 0$.

Theorem (Duchamp, Xie and Y.)

Order 4 proximity condition is satisfied if either

- [(a) Basically Nothing] $S_{\text{lin}}$ has a dual symmetry, or
- [(b) Basically Impossible] The curvature tensor from $f$ vanishes and $f$ and $\exp_f$ agree up to the 4-th order.

- Not surprising: symmetry improves proximity by one order
- Very surprising to us: No 4th order behavior of $f : TM \to M$ is required in (a)!
- Auslander & Markus (Ann. of Math, 55): If a manifold has a complete, torsion free, flat affine connection then it’s universal cover is $\mathbb{R}^n$. 
Even with time-symmetry, analysis and computation both suggest that order 5 prox. cond. is impossible for the single base-point scheme, unless the manifold is flat.
Result on approximation order: interpolatory case

Proximity Condition $\Rightarrow$ Approx. Order Equivalence

Theorem (Xie and Y.)

Assume that $S_{\text{lin}}$ (and hence also $S$) is interpolatory. If $S$ and $S_{\text{lin}}$ satisfy the order $k$ proximity condition, then if $S_{\text{lin}}$ has a $O(h^{k+1})$ accuracy order, so does $S$. 
Invariance Property of Proximity Condition

**Theorem (Xie and Y.)**

The proximity condition is insensitive to the choice of coordinates on the manifold.

Also shown in the same paper:

Intrinsic proximity condition $\iff$ Extrinsic proximity condition
Interest in approximation tools for manifold-valued data

Order $k$ Proximity condition $\Rightarrow C^k$ equivalence

Proximity condition $\Rightarrow$ Approximation order equivalence (for quasi-interpolatory schemes)
Surprises

- **Surprise I:** Difference between interpolatory and non-interpolatory cases
- **Surprise II:** Difference between single and two base-point schemes
- **Surprise III:** In single base-point scheme:
  - Choice of retraction matters
  - Time-symmetry matters
  - Space-symmetry does not matter (for the analysis, but desirable in practice)
  - Curvature tensor manifests itself and ‘everything stops’ at order 5.
- **Surprise IV:** For two base-point scheme, judicious choice of base-points gives arbitrary smoothness, nothing else matters.