Proof of Proposition 1 in [2]: minimizing $W_{Bobenko}$ would not give an approximation to a Clifford torus

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For every grid size $m$ and $n$, we define a family of triangulated torus, denoted by $T_{m,n,\varepsilon}$, $\varepsilon \in (0, \pi/2)$, with the following properties:

1. It has $mn$ vertices, $2mn$ triangles and all vertices have valence 6.
2. All vertices lie on a single sphere.
3. The diameter of the ‘tunnel’ of the torus goes to 0 as $\varepsilon \downarrow 0$.
4. The ‘tunnel’ of the torus is triangulated by $2n$ (long and skinny) triangles.

In details, it is defined as follows. Let $\Delta u = 2\pi/m$, $\Delta v = 2\pi/n$. For $i = 0, \ldots, m-1$, $j = 0, \ldots, n-1$, let

$$V_{i,j} = \begin{bmatrix} \sin(\tau(i\Delta u)) \cos(j\Delta v) \\ \sin(\tau(i\Delta u)) \sin(j\Delta v) \\ \cos(\tau(i\Delta u)) \end{bmatrix}, \quad \text{where } \tau(u) = \frac{(\pi - 2\varepsilon)u}{2\pi(1-1/m)} + \varepsilon. \quad (0.1)$$

Note that $\tau$ maps $[0, 2\pi(1-1/m)]$ to $[\varepsilon, \pi - \varepsilon]$. These $m \cdot n$ points on the unit sphere are the vertices of $T_{m,n,\varepsilon}$. For the triangulation, each $V_{i,j}$ is connected to the six neighbors $V_{i+1,j}$, $V_{i+1,j+1}$, $V_{i,j+1}$, $V_{i-1,j}$, $V_{i-1,j-1}$, $V_{i,j-1}$, forming also six triangular faces incident on the vertex. In above, the $+/-$ in the first and second indices are modulo $m$ and modulo $n$, respectively.

![Figure 1](image_url)  

Figure 1: Various spherical tori and a flattened torus based on a sphere inversion

Note that the $2n$ vertices $V_{0,j}$ and $V_{m-1,j}$, $j = 0, \ldots, n-1$ are the vertices closest to the north and south pole, respectively. While these two group of vertices are the furthest apart geometrically, they are connected and form a vertical tunnel of the torus with $2n$ long skinny triangles.

**Proposition 0.1.** For every grid size $(m, n)$, $W_{Bobenko}(T_{m,n,\varepsilon})$ decreases monotonically to $4\pi$ as $\varepsilon \downarrow 0$.

**Proof.** We divide the proof into three steps.

1° For any triangulated torus $T_{m,n,\varepsilon}$ defined above,

$$W_{Bobenko}(T_{m,n,\varepsilon}) = \frac{1}{2} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} W_{i,j}, \quad \text{where } W_{i,j} = \sum_{(i',j') \sim (i,j)} \beta(i',j') - 2\pi \quad (0.2)$$
and $\beta(i', j')$ is an angle formed by the circumscribed circles of the two triangles sharing the edge connecting $(i, j)$ to $(i', j')$ [1, Definition 1]. We then notice that

$$W_{i,j} = 0, \forall i \neq 0, m - 1,$$

since the six vertices around $(i, j)$ form a convex neighborhood lying on a common sphere [1, Proposition 1]. By symmetry, $W_{0,j}$ and $W_{m-1,j}$ share the same value for all $j = 0, \ldots, n - 1$. So we have

$$W_{Bobenko}(T_{m,n,\varepsilon}) = n W_{0,0}.$$

It remains to show that $W_{0,0}$ decreases monotonicity to $4\pi/n$ as $\varepsilon \downarrow 0$.

2° To simplify computation, we take advantage of the M"obius invariance of $W_{Bobenko}$ by applying a sphere inversion that maps the unit sphere to the $z = 0$ plane. Specifically, we invert about the sphere with radius $\sqrt{2}$ and centered at the north pole $[0, 0, 1]^T$ of the unit sphere. This maps the south pole of the unit sphere to the origin, and the north pole to infinity; and it turns our spherical torus $T_{m,n,\varepsilon}$ to a flattened torus. See Figure 1.

Among the 7 vertices $V_{0,0}, V_{0,1}, V_{-1,0}, V_{-1,-1}, V_{0,-1}, V_{1,0}, V_{1,1}$ contributing to $W_{0,0}$, $V_{0,0}, V_{0,1}, V_{0,-1}$ are close to the north pole, they are sphere inverted to points far away from the origin; $V_{-1,0}, V_{-1,-1}$ are close to the south pole, they are mapped to points close to the origin. The last two neighbors $V_{1,0}, V_{1,1}$ are at an approximately constant distance from the north pole: their common polar angle is uniformly larger than, and approaches, $\pi/(m - 1)$ as $\varepsilon \to 0$. Therefore, these 7 vertices are mapped to $V_i' \in \mathbb{R}^2$, $i = 0, 1, \ldots, 6$ (in the same cyclic order) with the form

$$V_0' = \rho_1(\varepsilon)[1, 0], \quad V_1' = \rho_1(\varepsilon)[c, s], \quad V_2' = \rho_2(\varepsilon)[1, 0], \quad V_3' = \rho_2(\varepsilon)[c, -s].$$

where

$$\rho_1(\varepsilon) \gg \rho_2(\varepsilon) \gg \varepsilon, \quad \text{as } \varepsilon \to 0. \quad \text{By scale invariance, } W_{0,0} \text{ depends on } \varepsilon \text{ through}$$

$$\varepsilon_1 := \rho_3(\varepsilon)/\rho_1(\varepsilon), \quad \text{and } \varepsilon_2 := \rho_2(\varepsilon)/\rho_1(\varepsilon).$$

Clearly, $\rho_1$ (resp. $\rho_2$) increases (resp. decreases) as $\varepsilon$ decreases. So $\varepsilon_2(\varepsilon)$ is monotonic increasing for $\varepsilon \in (0, \pi/2)$.

Below, we only need the facts that $1 > \varepsilon_1 > \varepsilon_2 > 0$, $\varepsilon_2$ is monotone in $\varepsilon$ and $\varepsilon_1, \varepsilon_2 \to 0$ as $\varepsilon \to 0$.

3° To calculate $W_{0,0} = \sum_{i=1}^6 \beta_i - 2\pi$ we analyze each angle $\beta_i$ between the circumcircles of the two triangles sharing the edge $e_i = V_iV_i'$, now thought of as functions of $(\varepsilon_1, \varepsilon_2)$.

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1 The approach also introduces a family of planar tori needed in another result.
We first notice that the two triangles $V_0'V_2'V_4'$, $V_0'V_3'V_4'$ sharing $e_3$ are co-cyclic because $V_0'V_2'V_3'V_4'$ form an isosceles quadrilateral. Therefore $\beta_3$ is either 0 or $\pi$. A closer inspection based on orientation (or simply applying the formula below) tells us that $\beta_3 = 0$. Similarly, $\beta_6 = 0$. Next, we show that

$$\lim_{\epsilon_1,\epsilon_2 \to 0} \beta_4 = \pi = \lim_{\epsilon_1,\epsilon_2 \to 0} \beta_4, \quad \lim_{\epsilon_1,\epsilon_2 \to 0} \beta_2 = \frac{2\pi}{n} = \lim_{\epsilon_1,\epsilon_2 \to 0} \beta_5. \quad (0.4)$$

These limits are not hard to see geometrically based on Figure 2; but to get the finer monotonicity result we resort to algebra and use the formula

$$\cos(\beta_i) = \frac{(A, C)\langle B, D \rangle - (A, B)\langle C, D \rangle - (B, C)\langle D, A \rangle}{\|A\|\|B\|\|C\|\|D\|},$$

where $A = V_0' - V_{i\oplus 1}'$, $B = V_{i\odot 1}' - V_0'$, $C = V_i' - V_{i\oplus 1}'$, $D = V_{i\odot 1}' - V_i'$. Here $i\oplus 1 = i + 1$ if $i < 6$, and 1 if $i = 6$, $i\odot 1 = i - 1$ if $i > 1$, and 6 if $i = 1$. By computation, we get $\cos(\beta_1) = \cos(\beta_4) = -\frac{1-\epsilon_1c-\epsilon_2c+\epsilon_1\epsilon_2}{\sqrt{1-2\epsilon_1c+\epsilon_1^2}\sqrt{1-2\epsilon_2c+\epsilon_2^2}}$. The limits (0.4) then follow immediately and we have proved that $\lim_{\epsilon_1,\epsilon_2 \to 0} W_{0,0}(\epsilon_1, \epsilon_2) = 4\pi/n$. To see that the convergence is monotone in the original $\epsilon$, write

$$W_{0,0}(\epsilon_1, \epsilon_2) = \cos^{-1}\left(\frac{c - 2\epsilon_1 + \epsilon_2^2}{1 - 2\epsilon_1c + \epsilon_1^2}\right) + \cos^{-1}\left(\frac{c - 2\epsilon_2 + \epsilon_2^2}{1 - 2\epsilon_2c + \epsilon_2^2}\right) - 2\cos^{-1}\left(\frac{1 - \epsilon_1c - \epsilon_2c + \epsilon_1\epsilon_2}{\sqrt{1 - 2\epsilon_1c + \epsilon_1^2}\sqrt{1 - 2\epsilon_2c + \epsilon_2^2}}\right).$$

From this, we get $\frac{\partial W_{0,0}}{\partial \epsilon_1} = 0$, $\frac{\partial W_{0,0}}{\partial \epsilon_2} = \frac{4\epsilon}{1 - 2\epsilon_2c + \epsilon_2^2} > 0$, when $1 \geq \epsilon_1 \geq \epsilon_2 \geq 0$. This shows $W_{0,0}(\epsilon_1, \epsilon_2)$ is independent of $\epsilon_1$, which also means $W_{0,0} = 2\cos^{-1}\left(\frac{c - 2\epsilon_2 + \epsilon_2^2}{1 - 2\epsilon_2c + \epsilon_2^2}\right)$, and is monotone in $\epsilon_2$. Combined with the monotonicity of $\epsilon_2(\epsilon)$, the proof is completed.

In the genus $g = 0$ case, as long as the connectivity is reasonable (see [1, Proposition 9]), the simplicial sphere is inscribable in a sphere and hence has a minimum $W_{\text{Bohonenk}}$-energy $4\pi$ (0 in Bohonenk’s definition of $W = \text{our definition of } W_{\text{Bohonenk}} - 4\pi(1 - g)$), the same as what one expects from the smooth setting.

Our result here shows that in the genus 1 case with even the perfectly regular connectivity, the infimum is never what one expects from the smooth setting. It also suggests, but does not prove, that the infimum is $4\pi$ regardless of $m$ and $n$, and that a minimizer does not exist as what we typically call a genus 1 PL surface. The latter, if proved, is another contrast to the smooth setting; cf. [3]. It is also possible to formulate and prove a generalization of this result for genus $g > 1$.

![Figure 3](image-url)

Figure 3: The minimization process rounds the 2- and 3-hole tori in (a) and (c) and closes up the holes, resulting in the empirical ‘infimumizers’ in (b) & (d), both with $W_{\text{Bohonenk}} \approx 4\pi$.

References
