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J. Math. Anal. Appl. 302 (2005) 201–216

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

On the regularity analysis of interpolatory Hermite subdivision schemes

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Received 11 April 2004

Available online 22 September 2004

Submitted by J. Lavery

Abstract

It is well known that the critical Hölder regularity of a subdivision schemes can typically be expressed in terms of the joint-spectral radius (JSR) of two operators restricted to a common finite-dimensional invariant subspace. In this article, we investigate interpolatory Hermite subdivision schemes in dimension one and specifically those with optimal accuracy orders. The latter include as special cases the well-known Lagrange interpolatory subdivision schemes by Deslauriers and Dubuc. We first show how to express the critical Hölder regularity of such a scheme in terms of the joint-spectral radius of a matrix pair $\{F_0, F_1\}$ given in a very explicit form. While the so-called finiteness conjecture for JSR is known to be not true in general, we conjecture that for such matrix pairs arising from Hermite interpolatory schemes of optimal accuracy orders a “strong finiteness conjecture” holds: $\rho(F_0, F_1) = \rho(F_0) = \rho(F_1)$. We prove that this conjecture is a consequence of another conjectured property of Hermite interpolatory schemes which, in turn, is connected to a kind of positivity property of matrix polynomials. We also prove these conjectures in certain new cases using both time and frequency domain arguments; our study here strongly suggests the existence of a notion of “positive definiteness” for non-Hermitian matrices.

Keywords: Hölder regularity; Subdivision scheme; Refinement equation; Hermite interpolation; Wavelets; Multiwavelets; Joint spectral radii; Matrix polynomial; Positivity; Fejer–Riesz factorization

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.07.051

1. Introduction

Subdivision algorithms are iterative methods for producing smooth curves and surfaces with a built-in multiresolution structure. They are now used in surface modeling in computer-aided geometric design. They are also intimately connected to wavelet bases and their associated fast filter bank algorithms. In the regular, one-dimensional setting, a subdivision scheme is given by a linear operator $S := S_a$ of the form

$$Sv = \sum_{\beta \in \mathbb{Z}} v(\beta) a(\alpha - 2\beta), \tag{1.1}$$

where $a \in [\ell^0(\mathbb{Z})]^{m \times m}$. Applying S to sequence v is the same as upsampling v by a factor of 2 followed by convolving the result with a. For computer-aided geometric design applications, we are particularly interested in vector subdivision schemes in which the components of each vector $S^n v(\alpha)$ measures a quantity with sensible geometric interpretation. Hermite subdivision schemes are such vector subdivision schemes. In dimension 2, they have direct applications in surface modeling [12–14,21,22]; in dimension 1, they are intimately connected to the moment-interpolating refinement schemes and multiwavelets by Donoho et al. [8].

The purpose of this note is two-fold.

- Firstly, we obtain detailed information—not directly provided by the existing theory for general subdivision scheme and refinement equation—about the convergence and Hölder regularity properties of interpolatory Hermite subdivision schemes in 1-D (Section 2), especially those with optimal accuracy order (Section 3). A basic tool that we use is that the (multiple knot) divided differences of the data generated by an interpolatory Hermite subdivision scheme follows another subdivision scheme, a result that we borrow from [9]. While this method can be can be thought of as a special case of the so-called factorization technique (see, e.g., [2,4]), being specific to Hermite subdivision the tailor-made method respects the symmetry properties of Hermite subdivision schemes more than the general factorization method, and allows us to determine the critical regularity of a Hermite subdivision scheme in terms of the joint spectral radius (JSR) of two matrices given in a very explicit form—these features of our analysis method are instrumental to the development that comes next.
- While JSR is in general difficult to compute, we conjecture that in the case of Hermite subdivision schemes with optimal accuracy order, the corresponding JSR satisfy $\rho(F_0, F_1) = \rho(F_0) = \rho(F_1)$, and hence are very easy to compute. We show in Proposition 3.2 how this conjecture follows from another "max-at-center" conjecture (Conjecture 3.1) of the subdivision scheme. We prove this "max-at-center" property in specific cases using a time domain argument, then in Section 4 we lay out a frequency domain argument which says that if a mathematical structure reminiscent of *positive definiteness*—but for matrices with a symmetry property different from being Hermitian—actually exists, then the "max-at-center" property, and hence the original conjecture, holds.

1.1. Interpolatory Hermite subdivision schemes

1-D interpolatory Hermite subdivision schemes behave in such a way that (see Definition 1.1) for every $v \in [\ell(\mathbb{Z})]^{1\times m}$, there is a C^{m-1} function f such that

$$D_n^{<m} f = S_a^n v, \quad \forall n \geqslant 0,$$

where $D_n^{<m} f$ is the (vector) sequence defined by

$$D_n^{<m} f(\alpha) := \left[f(x), 2^{-n} f'(x), \ldots, 2^{-n(m-1)} f^{(m-1)}(x)\right]\Big|_{x=2^{-n}\alpha}.$$

(One can also consider Hermite subdivision schemes of non-interpolatory type: i.e., $D_n^{<m} f \approx S_a^n v$, n large. See Fig. 1 and [14].)

By linearity and shift invariance of subdivision operators, it suffices to analyze the "impulse response" of the subdivision scheme:

Definition 1.1. Let $a_n := S_a^n a_0$, $a_0 = \delta I_{m\times m}$. We say that S_a is a convergent Hermite interpolatory subdivision scheme (of order $m-1$) if there exists $\phi = [\phi_0, \ldots, \phi_{m-1}]^T$, $\phi_i \in C^{m-1}(\mathbb{R})$, such that

$$2^{-nl} \phi_i^{(l)}(2^{-n}\alpha) = \big(a_n(\alpha)\big)_{i+1,l+1}, \quad \alpha \in \mathbb{Z},\ 0 \leqslant i, l < m,\ n = 0, 1, 2, \ldots. \tag{1.2}$$

In particular, ϕ is a Hermite interpolant, i.e., $\phi_i^{(l)}(\alpha) = \delta_{\alpha,0}\delta_{i,l}$.

The limit function vector ϕ, if exists, must be unique and satisfy the well-studied *vector refinement equation*

$$\phi = \sum_\alpha a(\alpha)\phi(2\cdot - \alpha).$$

This ϕ is a so-called refinable Hermite interpolant. From refinability and Hermite interpolation property of ϕ, we can derive that

$$a(2\alpha) = \operatorname{diag}\big(1, 2^{-1}, \ldots, 2^{-(m-1)}\big)\delta_{0,\alpha}.$$

A refinable Hermite interpolant gives rise to an interpolation operator: given Hermite data $\{v(\alpha)_l\colon 0 \leqslant l < m,\ \alpha \in \mathbb{Z}\}$ defined on the integers, the C^{m-1} function

$$\mathcal{I}_\phi v := \sum_\alpha v(\alpha)\phi(\cdot - \alpha)$$

satisfies $(\mathcal{I}_\phi v)^{(l)}(\alpha) = v(\alpha)_l$. Moreover, one can compute the Hermite samples of $\mathcal{I}_\phi v =: f$ digitally at any desired resolution via the subdivision operator: by (1.2)

$$D_n^{<m} f(\alpha) = (S^n v)(\alpha).$$

It is also important to note that the behavior of $\mathcal{I}_\phi v$ at one point depends only locally on the data v.

In the rest of the paper, we follow the notations defined in [13, Section 2.1].

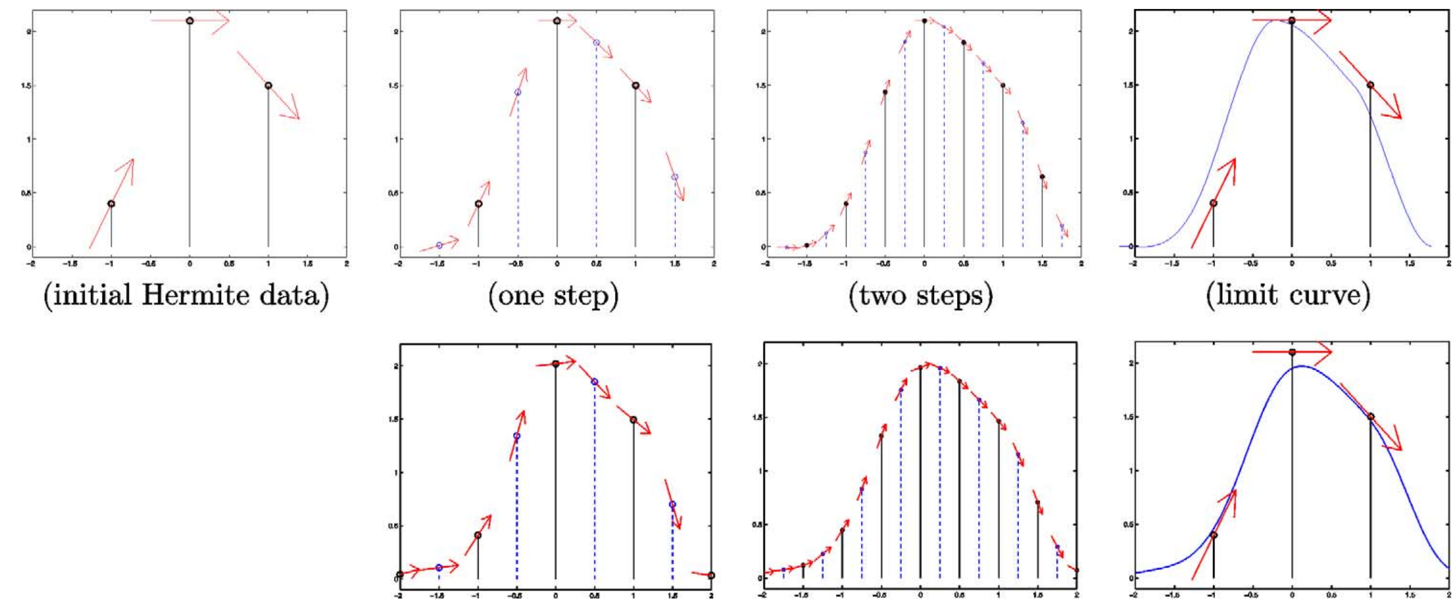

Fig. 1. 1-D interpolatory and non-interpolatory Hermite subdivision.

2. Convergence and smoothness analysis

There is now a large literature on the general theory of subdivision scheme and refinement equation. The following theorem is borrowed from the paper [13], which studies multivariate refinable Hermite interpolants.

Theorem 2.1 ([13], based on results from [3,16]). *Let a be a finitely supported mask with multiplicity m. The following are equivalent*:

(1) *There exists a unique refinable Hermite interpolant* $\phi \in [C^{m-1}(\mathbb{R})]^m$ *with mask* a.
(2) a *satisfies*: *for some* $k \geqslant m-1$,
   (i) $a(2\alpha) = \mathrm{diag}(1, 2^{-1}, \ldots, 2^{-(m-1)})\delta_{0,\alpha}$,
   (ii) Π_k *reproduction*: $S_a(D_0^{<m} p) = D_1^{<m} p$, $\forall p \in \Pi_k$,
   (iii)[1] *the eigenvalue condition of order* k: $J := \sum_\alpha a(\alpha)/2$ *has* 1 *as a simple eigenvalue and the other eigenvalues are of modulus less than* 2^{-k}, *and*
   (iv) $\max\{\lim_{n\to\infty} \|a_n * v\|_\infty^{1/n}: v \in H_k\} < 2^{-(m-1)}$ *where*

$$H_k := \left\{ v \in \left[\ell^0(\mathbb{Z}^s)\right]^{m\times 1}: \sum_{\alpha\in\mathbb{Z}^s} \left(D_0^{<m} p\right)(-\alpha)v(\alpha) = 0,\ \forall p \in \Pi_k(\mathbb{R}) \right\}. \quad (2.1)$$

In this case, the Hermite interpolatory subdivision scheme converges in the sense of Definition 1.1; *moreover, for* $k \geqslant m-1$, *if mask* a *does not reproduce* Π_{k+1}, *then the critical Hölder regularity of* ϕ *is given by*

$$\sup\left\{\nu: \phi \in [C^\nu]^{m\times 1}\right\} =: \nu_\infty(\phi) = -\log_2 \max\left\{ \lim_{n\to\infty} \|a_n * v\|_\infty^{1/n}: v \in H_k \right\}. \quad (2.2)$$

How can one compute the quantity in (2.2)? *What does the quantity in* (2.2) *really measure*? and *why is it related to the Hölder smoothness of* ϕ? To address these questions, we relate Theorem 2.1 to a method that Dyn and Levin had used to analyze Hermite interpolatory schemes [9]. The method in [9] is elementary and intuitive, but is very specific for 1-D Hermite interpolatory subdivision schemes; whereas Theorem 2.1 is based on the much more general but technical theory for refinement equations. This connection strengthens the result in [9] and is essential for the development in the next section.

The connection is in essence an observation on the structure of the space H_k, based on multiple-knot divided differences, which we recall as follows: for a smooth enough function f, define $[p_0]f := f(p_0)$ and, for $k > 0$ and $p_0 \leqslant \cdots \leqslant p_k$,

$$[p_0, \ldots, p_k]f := \begin{cases} \frac{1}{k!} f^{(k)}(p_0) & \text{if } p_0 = \cdots = p_k, \\ \frac{[p_1,\ldots,p_k]f - [p_0,\ldots,p_{k-1}]f}{p_k - p_0} & \text{if } p_0 \neq p_k. \end{cases} \quad (2.3)$$

[1] This condition can be dispensed with, cf. [10, Corollary 5.2].

We now define an operator $\tilde{\Delta}_k := \tilde{\Delta}_{k,m} : [\ell(\mathbb{Z})]^{1\times m} \to [\ell(\mathbb{Z})]^{1\times m}$ that, when $k \geqslant m-1$, maps a given sequence of Hermite data (defined on a regular grid) to their kth order divided differences, but scaled by (grid size)^k. Formally, for $v \in [\ell(\mathbb{Z})]^{1\times m}$, interpret v as

$$v(\alpha) = \left[f, hf', h^2 f'', \ldots, h^{m-1} f^{(m-1)}\right]\Big|_{h\alpha}$$

for some $h > 0$ and smooth function f, then formally apply (2.3) to get

$$\mathbb{R}^{1\times m} \ni (\tilde{\Delta}_k v)(\alpha) := \begin{cases} \left[\left(h^k[t_{\alpha+\lfloor l/m\rfloor}, \ldots, t_{\alpha+\lfloor (l+k)/m\rfloor}]f\right)_{0\leqslant l\leqslant k}, \right. \\ \quad \left.\left(h^l[\underbrace{t_\alpha, \ldots, t_\alpha}_{(l+1)t'_\alpha s}]f\right)_{k<l<m}\right] & \text{if } k < m-1, \\ \left[\left(h^k[t_{\alpha+\lfloor l/m\rfloor}, \ldots, t_{\alpha+\lfloor (l+k)/m\rfloor}]f\right)_{0\leqslant l<m}\right] & \text{if } k \geqslant m-1, \end{cases} \tag{2.4}$$

where $t_\alpha := h\alpha$. This definition makes sense: note that $\tilde{\Delta}_k v$ is independent of the hypothetical f and h. One can also extend the definition of $\tilde{\Delta}_k$ in a row-wise fashion to define an operator on $[\ell(\mathbb{Z})]^{m'\times m}$ for any m', and treat it as a bounded operator on $[\ell^p(\mathbb{Z})]^{m'\times m}$ for any $p \in [1, \infty]$. The operator Δ on $[\ell(\mathbb{Z})]^{m'\times m}$ is the usual (backward) differencing operator, defined by $\Delta v = v - v(\cdot - 1)$. Notice that $\tilde{\Delta}_{k,1} = \frac{1}{k!}\Delta^k := \frac{1}{k!}\Delta \circ \cdots \circ \Delta$.

Lemma 2.2. *If $\tilde{\Delta}_{k+1} u = 0$, then there exists $p \in \Pi_k$ such that $D_0^{<m} p = u$.*

Proof. It is obvious from the definition of divided difference that if $\tilde{\Delta}_s u = \tilde{\Delta}_s v$ and $(\tilde{\Delta}_{s-1} u(\alpha))_l = (\tilde{\Delta}_{s-1} v(\alpha))_l$ for some α and l, then $\tilde{\Delta}_{s-1} u = \tilde{\Delta}_{s-1} v$. Let

$$p := \sum_{d=0}^{k} \left(\tilde{\Delta}_d u(0)\right)_1 \prod_{l=0}^{d-1} \left(x - \lfloor l/m \rfloor\right).$$

This is the unique polynomial in Π_k such that $[\lfloor 0/m\rfloor, \ldots, \lfloor d/m\rfloor]p = (\tilde{\Delta}_d u(0))_1$ for $d = 0, \ldots, k$. But $\tilde{\Delta}_{k+1}(D_0^{<m} p) = 0 = \tilde{\Delta}_{k+1} u$, so $\tilde{\Delta}_s(D_0^{<m} p) = \tilde{\Delta}_s u$, for all $s = k, k-1, \ldots, 0$; so $u = D_0^{<m} p$. □

Theorem 2.3. *Let $D := \tilde{\Delta}_{k+1}(\delta I_{m\times m})$. Define $v_l \in [\ell^0(\mathbb{Z})]^{m\times 1}$ by $v_l(\alpha) =$ the lth column of $D(\alpha)$. Then H_k is spanned by $\{v_l(\cdot - \alpha)\colon 1 \leqslant l \leqslant m,\ \alpha \in \mathbb{Z}\}$.*

Proof. Note that $\tilde{\Delta}_{k+1} v = v * D$ for any $v \in [\ell(\mathbb{Z})]^{m'\times m}$. Each element $u \in [\ell(\mathbb{Z})]^{1\times m}$ defines a linear functional on $[\ell^0(\mathbb{Z})]^{m\times 1}$ via $\langle u, v\rangle := \sum_\alpha u(-\alpha)v(\alpha)$, $v \in [\ell^0(\mathbb{Z})]^{m\times 1}$ and that every linear functional on $[\ell^0(\mathbb{Z})]^{m\times 1}$ is of this form. Let $W := \operatorname{span}\{v_l(\cdot - \alpha)\colon 1 \leqslant l \leqslant m,\ \alpha \in \mathbb{Z}\}$. Since $\tilde{\Delta}_{k+1,m}(D_0^{<m} p) = 0$ for $p \in \Pi_k$, we have $W \subseteq H_k$. Assume $W \subsetneq H_k$, then $W^\perp \supsetneq H_k^\perp$ and there exists $u \in W^\perp \backslash H_k^\perp$. But $u \in W^\perp$ means $\tilde{\Delta}_{k+1} u = 0$ and, by Lemma 2.2, $u = D_0^{<m} p$ for some $p \in \Pi_k$. Therefore $u \in H_k$, a contradiction. Thus $W = H_k$. □

It is immediate from Theorem 2.3 that

$$\max_{v\in H_k}\lim_{n\to\infty}\|a_n*v\|_\infty^{1/n}=\lim_{n\to\infty}\|\tilde{\Delta}_{k+1}a_n\|_\infty^{1/n}$$
$$=\max_{w\in[\ell^0(\mathbb{Z})]^{1\times m}}\lim_{n\to\infty}\left\|\tilde{\Delta}_{k+1}S_a^n w\right\|_\infty^{1/n}. \tag{2.5}$$

Dyn and Levin had shown in [9, Theorem 1] that if the subdivision mask a is of Hermite interpolatory type and reproduces Π_k (see Theorem 2.1), then there exists a finitely supported subdivision mask $d\in[\ell^0(\mathbb{Z})]^{m\times m}$ such that

$$\tilde{\Delta}_{k+1}S_a^n w=S_d^n(\tilde{\Delta}_{k+1}w),\quad \forall w,\ \forall n. \tag{2.6}$$

Consequently, if we write $D_k:=\{\tilde{\Delta}_{k+1}w\colon\ w\in[\ell^0(\mathbb{Z})]^{1\times m}\}$,

$$\max_{v\in H_k}\lim_{n\to\infty}\|a_n*v\|_\infty^{1/n}=\max_{v\in D_k}\lim_{n\to\infty}\left\|S_d^n v\right\|_\infty^{1/n}. \tag{2.7}$$

Define $d_0=\delta I_{m\times m}$, $d_n:=S_d^n d_0$. We claim that

$$\max_{v\in D_k}\lim_{n\to\infty}\left\|S_d^n v\right\|_\infty^{1/n}=\lim_{n\to\infty}\|d_n\|_\infty^{1/n}=\max_{v\in[\ell^0(\mathbb{Z})]^{1\times m}}\lim_{n\to\infty}\left\|S_d^n v\right\|_\infty^{1/n}. \tag{2.8}$$

The second equality is obvious. Since $S_d^n v=(2^n\uparrow v)*d_n$, we have

$$\max_{v\in D_k}\lim_{n\to\infty}\left\|S_d^n v\right\|_\infty^{1/n}\leqslant\lim_{n\to\infty}\left(\left\|(2^n\uparrow v)\right\|_1\|d_n\|_\infty\right)^{1/n}=\lim_{n\to\infty}\|v\|_1^{1/n}\cdot\lim_{n\to\infty}\|d_n\|_\infty^{1/n}$$
$$\leqslant\lim_{n\to\infty}\|d_n\|_\infty^{1/n}.$$

On the other hand, notice that one can always find $w\in[\ell(\mathbb{Z})]^{m\times m}$ such that $\tilde{\Delta}_{k+1}w=\delta I_{m\times m}$; therefore there exists $w\in[\ell^0(\mathbb{Z})]^{m\times m}$ such that $v:=\tilde{\Delta}_{k+1}w$ agrees with $\delta I_{m\times m}$ on $[-N,N]$ for an arbitrarily given N. For large enough N, $S_d^n v$ coincide with d_n on the support of d_n, thus $\|S_d^n v\|_\infty\geqslant\|d_n\|_\infty$. Hence

$$\max_{v\in D_k}\lim_{n\to\infty}\left\|S_d^n v\right\|_\infty^{1/n}\geqslant\lim_{n\to\infty}\|d_n\|_\infty^{1/n}.$$

We have just proved the first equality in (2.8).

If $\|S_d\|_\infty$ denotes the operator norm of $S_d:[\ell^\infty(\mathbb{Z})]^{1\times m}\to[\ell^\infty(\mathbb{Z})]^{1\times m}$, then the quantity in (2.8) equals

$$\lim_{n\to\infty}\left\|S_d^n\right\|_\infty^{1/n}=\rho_\infty(S_d)=\inf_{n>0}\left\|S_d^n\right\|_\infty^{1/n}, \tag{2.9}$$

where $\rho_\infty(S_d)$ is the spectral radius of S_d when viewed as an element in the Banach algebra of all bounded operators acting on $[\ell^\infty(\mathbb{Z})]^{1\times m}$, and (2.9) is the well-known spectral radius formula [17, Chapter 18].

Putting together (2.5)–(2.8) and Theorem 2.1, we have

Theorem 2.4. *Let a be a Hermite interpolatory subdivision mask with multiplicity m which reproduces Π_k for some $k\geqslant m-1$ but does not reproduce Π_{k+1}, d be the subdivision mask in* (2.6) *and $d_n=S_d^n(\delta I_{m\times m})$. If $\nu:=-\log_2\lim_{n\to\infty}\|d_n\|_\infty^{1/n}>m-1$, then the Hermite interpolatory subdivision scheme converges to a refinable Hermite interpolant ϕ with mask a; moreover, $\nu_\infty(\phi)=\nu$.*

3. Hermite interpolatory schemes with optimal polynomial reproduction orders

As we have seen, polynomial reproduction is a necessary condition for smoothness, so for a given support of a it is natural to consider the mask with the highest possible polynomial reproduction order. This approach, however, in general does not lead to the smoothest possible scheme with the given support. See [11,24].

For given $m \geqslant 1$ and $L \geqslant 1$, by the well-posedness of Hermite interpolation, one can define an operator S on $[\ell(\mathbb{Z})]^{1\times m}$ such that

$$(Sv)(2\alpha) = v(\alpha)\,\mathrm{diag}\big(\big[1, 2^{-1}, \ldots, 2^{-(m-1)}\big]\big),$$
$$(Sv)(2\alpha+1) = \big[p, p', \ldots, p^{(m-1)}\big]\big|_{\alpha+1/2}\,\mathrm{diag}\big(\big[1, 2^{-1}, \ldots, 2^{-(m-1)}\big]\big),$$

where p is the unique polynomial in Π_{2mL-1} which satisfies $[p(\beta), p'(\beta), \ldots, p^{(m-1)}(\beta)] = v(\beta)$ for $\alpha - L < \beta \leqslant \alpha + L$. It is easy to see that S is exactly a Hermite-interpolatory subdivision operator, with a mask with support $[-(2L-1), (2L-1)]$; moreover, it reproduces Π_{2mL-1} but not Π_{2mL}, and that no scheme with that support can have a higher polynomial reproduction order. Such subdivision schemes had been studied by various authors.

Here we use the notations $a^{m,L}$ and $d^{m,L}$ to refer to the above Hermite interpolatory mask with optimal polynomial reproduction order and its difference mask satisfying (2.6) with $k = 2mL - 1$, i.e., $\tilde{\Delta}_{2mL} S^n_{a^{m,L}} w = S^n_{d^{m,L}}(\tilde{\Delta}_{2mL} w)$.

By symmetry properties of polynomials and the operator $\tilde{\Delta}_{2mL}$, one can see from rather straightforward calculations (see [23, Theorem 5.1] for details) that the support size and symmetry pattern of $a^{m,L}$ and $d^{m,L}$ are given in Table 1.

We now get back to convergence and smoothness analysis which, by Theorem 2.4, boils down to the quantity

$$\lim_{n\to\infty} \|d_n\|_\infty^{1/n} = \rho_\infty(S_{d^{m,L}}), \tag{3.1}$$

with $d_n = S^n_{d^{m,L}}(\delta I_{m\times m})$. But a subdivision operator acts *locally*, so the spectral radius of the infinite-dimensional operator $S_{d^{m,L}}$ equals to the *joint-spectral radius* of two finite matrices. Let $C_i := d^{m,L}(L-1-i)^T$, $i = 0, \ldots, 2(L-1)$, when $L > 1$, define the following two $m(2L-3) \times m(2L-3)$ matrices:

Table 1
Support and symmetry properties of $a^{m,L}$ and $d^{m,L}$

	Support	Symmetry
$a^{m,L}$	$[-(2L-1), (2L-1)]$	$a^{m,L}(-\alpha) = N a^{m,L}(\alpha) N$, $N = \mathrm{diag}((-1)^0, (-1)^1, \ldots, (-1)^{m-1})$
$d^{m,L}$	$[-(L-1), (L-1)]$	$d^{m,L}(-\alpha) = O_m d^{m,L}(\alpha) O_m$, $(O_m)_{i,j} = \delta_{i+j-m}$

$$F_0 = \begin{pmatrix} C_0 & C_2 & C_4 & \cdots & \cdots & \cdots \\ 0 & C_1 & C_3 & C_5 & \cdots & \cdots \\ 0 & C_0 & C_2 & C_4 & \cdots & \cdots \\ 0 & 0 & C_1 & C_3 & C_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$F_1 = \begin{pmatrix} C_1 & C_3 & C_5 & \ldots & \cdots & \cdots \\ C_0 & C_2 & C_4 & \ldots & \cdots & \cdots \\ 0 & C_1 & C_3 & C_5 & \cdots & \cdots \\ 0 & C_0 & C_2 & C_4 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.2}$$

When $L = 1$, define $F_0 = F_1 = C_0$. Then from the local action of $S_{d^{m,L}}$, one can see that

$$\max_{\varepsilon_i=0,1} \left\| \prod_{i=1}^{n} F_{\varepsilon_i} \right\|_\infty = \left\| S^n_{d^{m,L}} \right\|_\infty$$

and

$$\lim_{n\to\infty} \|d_n\|^{1/n} = \rho(S_{d^{m,L}}) = \rho(F_0, F_1) := \lim_{n\to\infty} \max_{\varepsilon_i=0,1} \left\| \prod_{i=1}^{n} F_{\varepsilon_i} \right\|^{1/n}. \tag{3.3}$$

The right-hand side of (3.3) is applicable to any matrix pair $\{F_0, F_1\}$ and is called the *joint-spectral radius* of the matrix pair. Joint-spectral radii are in general difficult to compute exactly, the so-called finiteness conjecture—disproved in [1]—speculates that for any two matrices A_0, A_1, there exists an $n \geqslant 1$ such that

$$\rho(A_0, A_1) = \max_{\varepsilon_i=0,1} \rho\left(\sum_{i=1}^{n} A_{\varepsilon_i} \right)^{1/n}. \tag{3.4}$$

Since the finiteness conjecture is not true in general, one may wonder under what condition is it true. Indeed, in several well-known examples arising from wavelets and subdivision schemes, one observes that (3.4) is true with $n = 1$. Here we conjecture that the same is true for the joint-spectral radii of those F_0, F_1 in (3.2). We shall show that this conjecture is implied by:

Conjecture 3.1. *For any $m \geqslant 1$, $L \geqslant 1$, let $d_n = S^n_{d^{m,L}}(\delta I_{m\times m})$, we have*

$$\|d_n\|_\infty = \left|d_n(0)\right|_\infty := \max_{i,j} \left| \left(d_n(0)\right)_{i,j} \right|, \quad \forall n = 1, 2, \ldots. \tag{3.5}$$

When $L > 1$, write $D_n := [d_{n-1}(-(L-2)), \ldots, d_{n-1}(L-2)]^T$, which is an array of size $m(2L-3) \times m$; there is a matrix F of size $m(2L-3) \times m(2L-3)$ such that $FD_n = D_{n+1}$ for all $n \geqslant 0$. This matrix is a common sub-matrix of F_0 and F_1. When $L = 1$, define $F = C_0$.

Proposition 3.2. *Assuming Conjecture* 3.1*, we have*

$$\lim_{n\to\infty} \|d_n\|_\infty^{1/n} = \lim_{n\to\infty} \left|d_n(0)\right|_\infty^{1/n} = \rho(F_0, F_1) = \rho(F_0) = \rho(F_1) = \rho(F). \tag{3.6}$$

Proof. When $L=1$, $F_0=F_1=F$ and the claim is trivial. For $L>1$, we have $F^n D_0 = D_n$ and

$$\max_{i,j}\left|(F^n D_0)_{i,j}\right| = \left|d_n(0)\right|_\infty. \tag{3.7}$$

Recall $D_0 = [0,\dots,0,I_{m\times m},0,\dots,0]^T \in \mathbb{R}^{m(2L-3)\times m}$. Let D_0' be a matrix with a block form similar to D_0, with an identity matrix as the only nonzero block but located not at the central position. The structure of subdivision implies that $F^n D_0'$ has the form $[d_{n-1}(-(L-2)+l),\dots,d_{n-1}(L-2+l)]^T$ for some $l\neq 0$. And (3.5) implies

$$\max_{i,j}\left|\left(F^n D_0'\right)_{i,j}\right| \leqslant \left|d_n(0)\right|_\infty. \tag{3.8}$$

But notice that the columns of D_0 and that of the different D_0' constitute the standard coordinate basis $\{e_l\}$ of $\mathbb{R}^{m(2L-3)}$, thus (3.7) and (3.8) can be rewritten as

$$\max_{e_l}\|F^n e_l\|_\infty = \left|d_n(0)\right|_\infty, \quad \forall n.$$

But $\max_{e_l}\|F^n e_l\|_\infty \asymp \|F^n\|_\infty$ as $n\to\infty$ and $\lim_{n\to\infty}\|F^n\|_\infty^{1/n} = \rho(F)$, so we have $\rho(F) = \lim_{n\to\infty}|d_n(0)|_\infty^{1/n}$.

With trivial modifications of the argument above, with F replaced by either F_0 or F_1, one shows that $\rho(F_i) = \lim_{n\to\infty}|d_n(0)|_\infty^{1/n}$ for $i=0,1$. □

3.1. *Case study*

We now discuss Conjecture 3.1 and its consequence (3.6). We consider three cases analytically, and then we give numerical evidences that the conjecture is true in general.

3.1.1. *$m=1$, L arbitrary*

The subdivision schemes in this case are the Lagrange interpolatory schemes by Deslauriers and Dubuc [7]; and the conjecture is well known to be true, as it is well known [7,18] that $a^{1,L}$, and hence also $d^{1,L}$, has a positive Fourier transform: $\widehat{a^{1,L}}(\omega) = \sum_k a^{1,L}(k)e^{-ik\omega} \geqslant 0$ and $\widehat{d^{1,L}}(\omega) = \sum_k d^{1,L}(k)e^{-ik\omega} \geqslant 0$, consequently, $\widehat{d_n}(\omega) = \prod_{j=1}^n \widehat{d^{1,L}}(2^{j-1}\omega) \geqslant 0$ and

$$\begin{aligned}\left|d_n(\alpha)\right| &= \frac{1}{2\pi}\left|\int_0^{2\pi}\widehat{d_n}(\omega)e^{i\alpha\omega}\,d\omega\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\widehat{d_n}(\omega)\right|\left|e^{i\alpha\omega}\right|d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi}\widehat{d_n}(\omega)\,d\omega = d_n(0).\end{aligned} \tag{3.9}$$

3.1.2. *$L=1$, m arbitrary*

In this case the conjecture is obviously true, since $d^{m,1}$, and hence also all d_n, is supported at the origin. In this case, we also know exactly the critical Hölder regularity of the subdivision scheme, since it produces spline functions of piecewise polynomial of degree $2m-1$ with C^{m-1} knots at the integers. As such, $\nu_\infty(\phi) = m$, and the quantity (3.6) equals 2^{-m}.

3.1.3. $L = 2$, m arbitrary

In this case $d^{m,2}$ is supported at $[-1, 0, 1]$. We offer a computational procedure to verify (3.6) based on a "short support trick":

Proposition 3.3. *If $d^{m,2}(0)$ is diagonalizable, and V is such that $V^{-1}d^{m,2}(0)V$ is diagonal and*

$$\left\|\left(V^{-1}d^{m,2}(-1)V\right)^T\right\|_\infty + \left\|\left(V^{-1}d^{m,2}(1)V\right)^T\right\|_\infty \leqslant \rho\left(d^{m,2}(0)\right), \tag{3.10}$$

then (3.6) *holds and the quantity in* (3.6) *equals also to* $\rho(d^{m,2}(0))$.

Proof. Consider the mask $\tilde{d}(\alpha) := V^{-1}d^{m,2}(\alpha)V$ and let $\tilde{d}_n := S^n_{\tilde{d}}(\delta I_{m\times m})$. Then it is clear that $\tilde{d}_n(\alpha) = V^{-1}d_n(\alpha)V$ for all n, and hence

$$\|\tilde{d}_n\|_\infty \asymp \|d_n\|_\infty, \quad n \to \infty, \quad \text{and} \quad \lim_{n\to\infty} \|\tilde{d}_n\|_\infty^{1/n} = \lim_{n\to\infty} \|d_n\|_\infty^{1/n}. \tag{3.11}$$

We now prove by induction that (3.10) implies:

$$\|\tilde{d}_n\|_\infty = \left|\tilde{d}_n(0)\right|_\infty = \rho\left(d^{m,2}(0)\right)^n. \tag{3.12}$$

It is clearly for $n = 0$; now assume that (3.12) holds for $n = N$. We have

$$\tilde{d}_{N+1}(2\alpha) = \tilde{d}_N(\alpha)\tilde{d}(0), \quad \tilde{d}_{N+1}(2\alpha+1) = \tilde{d}_N(\alpha)\tilde{d}(1) + \tilde{d}_N(\alpha+1)\tilde{d}(-1).$$

Since $\tilde{d}(0)$ is diagonal,

$$\tilde{d}_n(0) = \tilde{d}(0)^n = \rho\left(d^{m,2}(0)\right)^n. \tag{3.13}$$

Note that for square matrices A, B, $|AB|_\infty \leqslant |A|_\infty \|B^T\|_\infty$. By induction hypothesis, we have

$$|\tilde{d}_{N+1,2\alpha}|_\infty = \left|\tilde{d}_N(\alpha)\tilde{d}(0)\right|_\infty \leqslant \left|\tilde{d}_N(\alpha)\right|_\infty \left\|\tilde{d}(0)^T\right\|_\infty \leqslant \rho\left(d^{m,2}(0)\right)^{N+1}, \tag{3.14}$$

$$|\tilde{d}_{N+1,2\alpha+1}|_\infty \leqslant \left|\tilde{d}_N(\alpha)\right|_\infty \left\|\tilde{d}(1)^T\right\|_\infty + \left|\tilde{d}_N(\alpha+1)\right|_\infty \left\|\tilde{d}(-1)^T\right\|_\infty$$

$$\leqslant \left|\tilde{d}_N(0)\right|_\infty \left(\left\|\tilde{d}(1)^T\right\|_\infty + \left\|\tilde{d}(-1)^T\right\|_\infty\right) \tag{3.15}$$

$$\leqslant \rho\left(d^{m,2}(0)\right)^N \rho\left(d^{m,2}(0)\right) = \rho\left(d^{m,2}(0)\right)^{N+1}. \tag{3.16}$$

This completes the proof of (3.12). Indeed, (3.12) is essentially Conjecture 3.1 with $d^{m,2}$ replaced by the transformed mask $\tilde{d}$, thus by the proof of Proposition 3.2, we have (3.6) with d_n, F and F_i modified by certain similarity transforms based on V. But such similarity transforms do not change the spectral quantities involved in (3.6) (e.g., (3.11)), thus (3.6) holds. □

Examples. $m = 1$:

$$a^{1,2} = (\ldots, 0, -1/16, 0, 9/16, 1, 9/16, 0, -1/16, 0, \ldots)$$

and

$$d^{1,2} = 2^{-4}(\ldots, 0, -1, 4, -1, 0, \ldots).$$

The condition (3.10) is satisfied and $\nu_\infty(\phi) = -\log_2(2^{-2}) = 2$, which recovers the result from [7]. In this case, the proof of Proposition 3.3 offers a time-domain proof for Conjecture 3.1 as opposed to the frequency-domain argument in (3.9).

$m = 2$:

$$d^{2,2}(-1) = 2^{-11}\begin{pmatrix} 5 & -61 \\ -3 & 43 \end{pmatrix}, \qquad d^{2,2}(0) = 2^{-7}\begin{pmatrix} 9 & -7 \\ -7 & 9 \end{pmatrix},$$
$$d^{2,2}(1) = 2^{-11}\begin{pmatrix} 43 & -3 \\ -61 & 5 \end{pmatrix}. \tag{3.17}$$

$d^{2,2}(0)$ is diagonalized by $V = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, $V^{-1}d^{2,2}(0)V = \mathrm{diag}(1/8, 1/64)$ and

$$\left\| \left(V^{-1}d^{2,2}(-1)V\right)^T \right\|_\infty + \left\| \left(V^{-1}d^{2,2}(1)V\right)^T \right\|_\infty = 33/512 < 1/8 = \rho\left(d^{2,2}(0)\right).$$

Thus condition (3.10) holds and $\nu_\infty(\phi) = -\log_2(1/8) = 3$.

$m = 3$:

$$d^{3,2}(-1) = \frac{1}{2^{18}}\begin{pmatrix} -25 & 115 & -3331 \\ 15 & -77 & 2581 \\ -9 & 51 & -1939 \end{pmatrix},$$
$$d^{3,2}(0) = \frac{1}{2^{13}}\begin{pmatrix} 207 & -201 & 143 \\ -177 & 215 & -177 \\ 143 & -201 & 207 \end{pmatrix},$$

$d^{3,2}(-1) = O_3 d^{3,2}(1) O_3$. One can verify that (3.10) holds and $\nu_\infty(\phi) = -\log_2\left(\frac{565+3\sqrt{33649}}{16384}\right) = 3.8768\ldots.$

Remark. The "short support trick" used in Proposition 3.3 has also been used in other contexts:

Irregular grid subdivision. In [6] Daubechies et al. study a scalar interpolatory subdivision scheme corresponding to $m = 1$ and $L = 2$ in this article but on *irregular* multigrid; they prove that as long as the irregularity of the grids is controlled in some way then the critical Hölder regularity of the limit function is *exactly* the same as that in the uniform grid case. They conjecture that the same phenomenon should hold for $L > 2$, but their method of proof for the case of $L = 2$ depends heavily on a trick very similar to that used in Proposition 3.3. It is natural to further conjecture that the same phenomenon holds for irregular grid Hermite subdivision schemes of any order $m - 1 \geqslant 0$ and support size $L \geqslant 1$, and the case of $L = 1$ and m arbitrary can in principle be verified by the same short support trick used in Proposition 3.3.

Nonlinear subdivision. A related short support trick is used in [19,20] to show that the critical Hölder regularity of a certain nonlinear (scalar) subdivision scheme with short support is the same as that of an approximating linear subdivision scheme.

Table 2
Lower and upper bounds of $\nu_\infty(\phi)$

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
$m=1$	1.0000/1.0000	2.0000/2.0000	2.8094/2.8301	3.5024/3.5511	4.1101/4.1936
$m=2$	2.0000/2.0000	3.0000/3.0000	3.6140/3.6173	3.9384/3.9416	4.1273/4.1286
$m=3$	2.9415/3.0000	3.8620/3.8768	4.4880/4.4963	4.9518/4.9646	5.3301/5.3520

3.1.4. Computational evidences

Besides the provable cases presented in the last three sections, there are also numerical evidences which strongly suggest that (3.6) and (3.5) are true for all m and L. It is well known that for any matrix pair F_0, F_1 the joint spectral radius $\rho(F_0, F_1)$ can be bounded as follows:

$$\max\bigl(\rho(F_0), \rho(F_1)\bigr) \leqslant \rho(F_0, F_1) \leqslant \max_{\epsilon_i=0,1} \left\| \prod_{i=1}^{j} F_{\epsilon_i} \right\|^{1/j}, \tag{3.18}$$

here the upper bound can be applied to any $j \geqslant 0$ and any matrix norm $\|\cdot\|$, and it converges to the joint spectral radius as $j \to \infty$. For the matrix pair arising from interpolatory Hermite subdivision schemes, we had computed the lower and upper bounds with $j = 10$ and the infinity norm in the cases of $m = 1, 2, 3$ and $L = 1, \ldots, 5$; and we observed that the upper and lower bounds agree up to 2 significant digits. (It is a known experience, and a provable fact in certain cases, that the convergence of the upper bound is very slow, typically at the rate of $O(1/j)$; yet the complexity for computing the bound increases as $O(2^j)$, so it is quite an inefficient method for estimating the joint spectral radius.) In virtue of (3.18), these numerical results suggest that (3.6) is true in general.

By Theorem 2.4 and (3.3), if $-\log_2 \rho(F_0, F_1) > m - 1$, then

$$\nu_\infty(\phi) = -\log_2 \rho(F_0, F_1),$$

where F_0 and F_1 are given by (3.2). Thus a lower (respectively upper) bound for $\rho(F_0, F_1)$ gives an upper (respectively lower) bound for $\nu_\infty(\phi)$. We report in Table 2 such bounds for $\nu_\infty(\phi)$ based on (3.18) with $j = 10$ and the infinity norm.

4. Open question: positivity for matrix polynomial?

This article suggests the following open problems.

[I] For finitely supported matrix sequences with symmetry property $d(-\alpha) = O_m d(\alpha) \times O_m$, identify a property on the matrix trigonometrical polynomial

$$\hat{d}(\omega) := \sum_\alpha d(\alpha) e^{-i\alpha\omega}$$

which generalizes $\hat{d}(\omega) \geqslant 0$ in the $m = 1$ case in such a way that any d with such a property would satisfy either (3.5) or (3.6).

[II] Prove that $d^{m,L}$ possesses the sought-for property for any (m, L).

4.1. Dissecting problem [I]

This final section should be accessible to anyone interested in matrix analysis, as the presentation is completely independent to the analysis of subdivision schemes.

In the scalar case $m = 1$, the key properties of scalar sequence d that lead to the desired conclusion (3.5) or (3.6) are the followings:

(p1) (Realness) The sequence d is symmetric about $\alpha = 0$, so that $\hat{d}(\omega)$ is real for all $\omega \in \mathbb{R}$.

(p2) (Positivity) Not only is $\hat{d}(\omega)$ real, but also that $\hat{d}(\omega) \geqslant 0$ for all ω.

(m) ("Max-at-center") By (p1) and (p2) and the simple argument in (3.9), we have $\max_\alpha |d(\alpha)| = d(0)$.

(c) (Closedness) The positivity property (p1)–(p2) is closed under the process of subdivision, i.e., if $d(\omega)$ is positive for all ω, then so is $d(2\omega)d(\omega)$. Indeed positivity is closed under any dilation and multiplication.

At this point, we have only a meager understanding of the possible generalization of the above properties to the matrix case. We report the following observations and speculations.

(P1) The symmetry condition $O_m d(\alpha) O_m = d(-\alpha)$ implies that

$$O_m \hat{d}(\omega) O_m = \overline{\hat{d}(\omega)}. \tag{4.1}$$

(Here the bar converts the operand matrix to the matrix with entries equal to the conjugate of the corresponding entries of the operand matrix, i.e., $\overline{M}_{i,j} = \overline{M_{i,j}}$.) The above property implies that, for each $\omega \in \mathbb{R}$, the eigenvalues of $\hat{d}(\omega)$ come in complex conjugate pair.

(P2) The author observed *empirically* the rather remarkable property that for any $d = d^{m,L}$ arising from 1-D Hermite interpolatory subdivision schemes,

$$\bigcup_{\omega \in \mathbb{R}} \sigma\bigl(\hat{d}(\omega)\bigr) \subset \mathbb{R}^+, \tag{4.2}$$

i.e., for any $\omega \in \mathbb{R}$, all the eigenvalues of $\hat{d}(\omega)$ are real and positive. For example, in the case of $(m, L) = (2, 2)$, by (3.17), the two eigenvalues of $\widehat{d^{2,2}}(\omega)$ are

$$\frac{9}{128} + \frac{3}{128}\cos(\omega) \pm \frac{1}{256}\sqrt{243 + 224\cos(\omega) + 17\cos(2\omega)} > 0.$$

(M) While (P1)–(P2) may sound like a natural generalization of (p1)–(p2) in the scalar case, they do not directly imply $\max_\alpha |d(\alpha)|_\infty = |d(0)|_\infty$ or any related "max-at-center" property of the matrix sequence d.

However, if there exists a linear map $G : \mathbb{C}^{m\times m} \to \mathbb{C}$ such that

$$G\bigl(\hat{d}(\omega)\bigr) \geqslant 0, \quad \forall \omega \in \mathbb{R}, \tag{4.3}$$

then the argument in (3.9) can be carried over to the matrix case:

$$\begin{aligned}
\big|G\big(d(\alpha)\big)\big| &= \left|G\left(\frac{1}{2\pi}\int_0^{2\pi}\hat{d}(\omega)e^{i\alpha\omega}\,d\omega\right)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}G\big(\hat{d}(\omega)\big)e^{i\alpha\omega}\,d\omega\right| \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\big|G\big(\hat{d}(\omega)\big)e^{i\alpha\omega}\big|\,d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi}G\big(\hat{d}(\omega)\big)\,d\omega \quad \text{(by (4.3))} \\
&= G\big(d(0)\big).
\end{aligned}$$

This is then a kind of "max-at-center" property.

So we further speculate that there is perhaps an interesting link between (4.3) and the spectral property (4.2); indeed it is fairly tempting to speculate such a link due to the following consideration: if $\hat{d}(\omega)$ *were* Hermitian, then (4.2) boils down to the usual concept of positive semi-definiteness, and the linear map $G_u : \mathbb{C}^{m\times m} \to \mathbb{C}$ defined by

$$G_u(M) := u^* M u \tag{4.4}$$

satisfies (4.3) for any vector $u \in \mathbb{C}^m$. However, our $\hat{d}(\omega)$'s are not Hermitian but satisfy (4.1) instead; on the other hand there could be linear maps not of the form (4.4) that does the job.

(C) (P1)–(P2) are clearly closed under dilation, but, without further restriction, may not be closed under multiplication.

5. Concluding remarks

The conjectures (3.5) and the implied consequence (3.6) were first made by the author in 1996, reported in the Conference on Theory and Applications of Multiwavelets (Sam Houston State University, Huntsville, Texas, March 20–22, 1997) and documented in the unpublished [23]. The sought-for positivity structure is also related to a matrix version of Feyer–Riesz factorization (a.k.a. spectral factorization), which has application in various filter design problems in signal processing (see, e.g., work in wavelet filter design [5,15]).

The observations made in this section, together with the computational and analytical evidences pointing to the validity of Conjecture 3.1, all seem to suggest that there is a useful notion of "generalized positive definiteness" for non-Hermitian matrices either waiting to be discovered or, if already known elsewhere, waiting to be connected to the problems identified in this article.

Acknowledgments

The author thanks Jeff Geronimo, Douglas Hardin, and Hugo Woerdeman for discussions. Special thanks go to Chi-Kwong Li for extensive discussions on the matrix analysis problem identified in this paper.

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