SIAM J. NUMER. ANAL.
Vol. 42, No. 3, pp. 997–1017

CONTINUOUS M-ESTIMATORS AND THEIR INTERPOLATION BY POLYNOMIALS*

JONG-SHI PANG† AND THOMAS P.-Y. YU‡

Abstract. Replacing the *median* by a general *M-estimator*, we construct in this paper a host of variants of the robust nonlinear pyramid transforms proposed by Donoho and Yu [*SIAM J. Math. Anal.*, 31 (2000) pp. 1030–1061]. Some of these new variants are more amenable to numerical implementations with provable properties when compared to the Donoho–Yu median-based pyramid transforms. At the crux of this generalized construction is the following result: the inverse problem of interpolating a univariate polynomial of degree n with $n+1$ prescribed values for any given continuous M-estimator on $n+1$ nonoverlapping intervals is a well-posed procedure. While the proof of this result is nonconstructive, we study the use of Newton methods for constructing such a polynomial interpolant and report numerical results in some test cases.

Key words. median, M-estimator, polynomial interpolation, Newton's method, convex analysis, nonlinear pyramid transforms, wavelets

AMS subject classifications. 58C15, 65D05, 65D10, 65T60, 34A34, 37L65

DOI. 10.1137/S0036142902412221

1. Introduction.

Donoho and Yu introduced in [3] a family of nonlinear pyramid transforms of signals which is reminiscent of linear biorthogonal wavelet transforms but has the advantage of being robust against non-Gaussian noise. Underlying their construction are the concept of measuring the median values of a signal over intervals at multiple scales and the accurate prediction of fine-scale median values from coarse-scale ones based on polynomial interpolation. The latter polynomial interpolation procedure, shown to be well posed by Goodman and Yu using a nonconstructive argument [6], involves the solution of a *nonsmooth* system of equations for which no existing equation solvers are shown to be convergent.

Instead of directly attacking the computationally challenging median interpolation problem, we introduce in this paper a broad class of "continuous M-estimators" that include the continuous median as a special case. These M-estimators naturally lead to new classes of nonlinear pyramid transforms that extend the original family introduced by Donoho and Yu, which is based on the continuous median. Extending the result of Goodman and Yu, we demonstrate that the interpolation problem of the continuous M-estimators by polynomials is also well posed. With the use of a "smooth kernel," the latter interpolation problem is equivalent to solving a system of differentiable equations, which can be accomplished by, say, a stabilized Newton method that can be demonstrated to be globally and quadratically convergent. In turn, this implies that the pyramid transforms can be implemented using a broad class of robust statistics with the aid of a provably convergent Newton method.

*Received by the editors July 25, 2002; accepted for publication (in revised form) December 22, 2003; published electronically July 14, 2004. This research was initiated during the second author's visit to The Johns Hopkins University in March 2002.

http://www.siam.org/journals/sinum/42-3/41222.html

†Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180-3590 and Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, MD 21218-2682 (pangj@rpi.edu). The research of this author was partially supported by the National Science Foundation under grant CCR-0098013.

‡Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180-3590 (yut@rpi.edu). The research of this author was partially supported by a NSF CAREER Award (CCR 9984501).

In summary, the main contribution of our work is twofold: first, based on the class of continuous M-estimators, we have significantly broadened the original Donoho–Yu family of nonlinear pyramid transforms; more importantly, a large class of such transforms can be implemented by solving *smooth* systems of nonlinear equations by provably convergent numerical methods.

2. Continuous M-estimators. We begin with a brief review of the median value of a continuous function $f : \mathbb{R} \to \mathbb{R}$ on an interval I and the associated interpolation problem. Specifically, the latter value is defined as

$$\text{med}(f|I) \;:=\; \underset{m\in\mathbb{R}}{\arg\min} \int_I |\, f(t) - m \,|\, dt. \tag{2.1}$$

A key step in the Donoho–Yu proposal for accurately predicting fine-scale medians from coarse-scale ones [3] (see section 5 for more details) requires the inversion of the map

$$M \;:\; \Pi_n \;\to\; \mathbb{R}^{n+1}, \quad p \mapsto (\,\text{med}(p|I_i)\,)_{i=0}^n, \tag{2.2}$$

where I_i, $i = 0, \dots, n$, are nonoverlapping intervals and Π_n is the $(n+1)$-dimensional vector space of polynomials of degree not exceeding n. While this nonlinear map is known to be a homeomorphism [6], implying that M^{-1} exists, computing $M^{-1}(a)$ for a given vector a is nevertheless not an easy task. In general, we have to resort to numerical methods. (In the case of $n = 2$, closed-form formulas for M^{-1} can be found in [3, section 2].) Newton's method applied to the system of nonlinear equations $M(p) = a$ is a prime candidate for this task. Nevertheless, since the map M is not everywhere differentiable, a classical Newton method for smooth systems (such as the ones in [2, 11]) is therefore not applicable. Although there exist provably convergent Newton methods for "semismooth" systems (see [5, Chapters 7 and 8] and the references therein), it is an open problem at this time whether M in (2.2) is semismooth in general.

Partly to broaden the continuous median and partly to alleviate the computational difficulty with inverting the nondifferentiable map M in (2.2), this paper introduces the class of continuous M-estimators defined with respect to triples (K, f, ρ) satisfying the following blanket specifications: (a) K is a solid, compact, connected set whose boundary, denoted ∂K, has measure zero; (b) f is a real-valued continuous function defined on K; and (c) $\rho : \mathbb{R} \to \mathbb{R}_+$ is a convex function with a unique minimizer at zero, and $\rho(0) = 0$. The solidness of K implies that its interior, denoted int K, is nonempty; hence K has positive measure. The convexity of ρ implies its continuity.

Corresponding to such a triple (K, f, ρ), we define the minimand $\theta(\cdot; K, f, \rho)$: $\mathbb{R} \to \mathbb{R}_+$ and the *continuous M-estimator* $m(f; K, \rho)$ as follows:

$$\begin{aligned} \theta(m; K, f, \rho) \;&:=\; \int_K \rho(f(x) - m)\, dx, \qquad m \in \mathbb{R}, \\ m(f; K, \rho) \;&:=\; \underset{m\in\mathbb{R}}{\arg\min}\, \theta(m; K, f, \rho). \end{aligned}$$

When the pair (K, ρ) is clear from the context, we write $m(f)$ for $m(f; K, \rho)$. Implicit in the above definition of $m(f; K, \rho)$ is the assertion that this is a well-defined quantity (i.e., it exists and is unique); this will be justified in Theorem 2.2. Throughout the paper, we let $C(K)$ denote the set of continuous real-valued functions defined on K.

Before establishing properties of the continuous M-estimator, we give several choices of the convex function ρ. Among these choices all are continuously differentiable except the absolute value function and the third one, which are both piecewise linear. The nomenclature “continuous M-estimator” is coined as a generalization of the well-known robust Huber M-estimator [7], which is recovered from the fourth function.

1. $\rho(t) = |t|^p$ for $p \geq 1$. For $p = 1$, the resulting $m(f; K, \rho)$ is the continuous median of f on K:

$$\text{med}(f|K) \,=\, \arg\min_{m} \int_K |\, f(x) - m \,|\, dx,$$

generalizing the case of an interval K. For $p = 2$, the resulting $m(f; K, \rho)$ is the average of f on K:

$$\text{ave}(f|K) = \frac{1}{\text{meas } K} \int_K f(x)\, dx,$$

where “meas” is the abbreviation of “measure.”

2. $\rho_\varepsilon(t) = \sqrt{t^2 + \varepsilon^2} - \varepsilon$ for some $\varepsilon > 0$; this function ρ_ε is a strictly convex, C^∞ approximation to the absolute-value function $|\cdot|$. As we will prove in Proposition 4.4, $m(f; K, \rho_\varepsilon)$ converges to $\text{med}(f|K)$ as $\varepsilon \downarrow 0$. Consequently, for $\varepsilon > 0$ sufficiently small, we expect $m(f; K, \rho_\varepsilon)$ to be as robust against outliers as the standard median. (A rigorous quantification of this claim is in the scope of robust statistics, which we will not get into in this paper.)

3. $\rho(t) = \alpha \max(t, 0) + (1 - \alpha) \max(-t, 0)$ for $\alpha \in (0, 1)$. The resulting $m(f; K, \rho)$ measures the continuous α-quantile of the function f over K, i.e., the unique value m, such that

$$\text{meas}\{t \in K : f(t) \geq m\} \leq \alpha \text{ meas } K \quad \text{and} \quad \text{meas}\{t \in K : f(t) \leq m\} \leq (1-\alpha)\text{meas } K.$$

4. For a given $c > 0$,

$$\rho(t) = \begin{cases} \frac{1}{2}\, t^2 & \text{if } |\, t \,| \,\leq\, c, \\ c\,|\, t \,| - \frac{1}{2}\, c^2 & \text{if } |\, t \,| \,>\, c. \end{cases} \tag{2.3}$$

The resulting $m(f; K, \rho)$ is a continuous version of the Huber estimator for discrete empirical data.

2.1. Basic properties. We begin our study of the continuous M-estimator by stating the following result that summarizes several basic properties of the minimand $\theta(\cdot; K, f, \rho)$ and shows in particular that this function inherits many properties of ρ. In the proof of this and other results, we will freely use known properties of convex functions, which can all be found in the classic treatise by Rockafellar [10].

PROPOSITION 2.1. *Let the triple (K, f, ρ) satisfy the blanket assumptions. The following statements hold for the function $\theta := \theta(\cdot; K, \rho, f)$.*

(a) *θ is convex (thus continuous) and coercive on $\mathbb{R}$; coercivity means*

$$\lim_{|m| \to \infty} \theta(m) \,=\, \infty.$$

(b) *If ρ is strictly convex, then so is θ.*

(c) *The right and left derivatives of* θ *are equal to, respectively,*

$$\theta'_+(m) \;:=\; \lim_{h\to 0+} \frac{\varphi(t+h)-\varphi(t)}{h} \;=\; -\int_K \rho'_-(f(x)-m)\,dx,$$

$$\theta'_-(m) \;:=\; \lim_{h\to 0+} \frac{\varphi(t)-\varphi(t-h)}{h} \;=\; -\int_K \rho'_+(f(x)-m)\,dx.$$

(d) *If* ρ *is* k*-times continuously differentiable on* $\mathbb{R}$ *for some* $k \geq 1$*, then so is* θ.
(e) *If* ρ *is twice differentiable and* $\rho''(t) > 0$ *for all* $t \in \mathbb{R}$*, then so is* θ.
(f) θ *is differentiable at* m *if and only if the set* Ω_m *has measure zero, where*

$$\Omega_m := \{\, x \in K : \rho \text{ is not differentiable at } f(x)-m \,\}.$$

Proof. The convexity of θ follows from that of ρ; the coercivity of θ follows from the inequalities

$$\liminf_{|m|\to\infty} \frac{\theta(m)}{|m|} \;\geq\; \min(\rho(1),\rho(-1)) \liminf_{|m|\to\infty} \int_K \frac{|f(x)-m|}{|m|}\,dx \;>\; 0.$$

Since K has positive measure, (b) is obvious. To prove (c), let $h \in (0,1]$. We have

$$\frac{\theta(m+h)-\theta(m)}{h} \;=\; \int_K \left[\frac{\rho(f(x)-m-h)-\rho(f(x)-m)}{h} \right] dx.$$

Since $\rho'_-(t)$ exists for all t, by the compactness of K, it follows that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in K$ and all $h \in (0,\delta]$,

$$\left| \frac{\rho(f(x)-m-h)-\rho(f(x)-m)}{h} \right| \;\leq\; |\,\rho'_-(f(x)-m)\,| + \varepsilon.$$

Since the right-hand side, as a function of x, is clearly integrable on K, it follows by the dominated convergence theorem that

$$\theta'_+(m) \;=\; \int_K -\rho'_-(f(x)-m)\,dx.$$

Similarly, we can establish the desired formula for the left derivative of θ at m. The differentiability of θ in parts (d) and (e) can be proved easily. The positivity of θ'' in part (e) follows from the formula

$$\theta''(m) \;=\; \int_K \rho''(f(x)-m)\,dx. \tag{2.4}$$

Finally, it is clear that if the set Ω_m has measure zero, then

$$\theta'_+(m) \;=\; \theta'_-(m) \;=\; -\int_{\mathcal{D}_m} \rho'(f(x)-m)\,dx,$$

where $\mathcal{D}_m$ is the complement of Ω_m in K. Conversely, suppose that Ω_m has positive measure. Since the set of nondifferentiable points of ρ is countable, it follows that there must exist a $t_0 \in \mathbb{R}$ such that the set

$$\Lambda := \{\, x \in K : f(x)-m = t_0 \,\}$$

has positive measure. Since $\rho'_+(f(x)-m) \geq \rho'_-(f(x)-m)$ for all $x \in K$, it follows that

$$\int_K [\,\rho'_+(f(x)-m) - \rho'_-(f(x)-m)\,]\,dx \;\geq\; (\,\rho'_+(t_0) - \rho'_-(t_0)\,)\,\text{meas}\,\Lambda.$$

Since the right-hand side is positive, it follows that $\theta'_-(m) > \theta'_+(m)$; (f) therefore holds. ◻

The next result summarizes various properties of the continuous M-estimator $m(f;K,\rho)$. Part (a) asserts the well-definedness of this minimizer and gives its variational characterization; part (b) says that this minimizer must belong to the range $f(\text{int}\ K)$; part (c) is a technical property that will be used subsequently.

THEOREM 2.2. *Let (K,ρ,f) satisfy the blanket assumptions. The following statements are valid.*

(a) *$m(f;K,\rho)$ exists and is unique; it is the unique scalar $\bar{m}$ satisfying*

$$\int_K \rho'_+(f(x)-\bar{m})\,dx \;\geq\; 0 \;\geq\; \int_K \rho'_-(f(x)-\bar{m})\,dx.$$

(b) *There exists $\bar{x} \in \text{int}\ K$ such that $f(\bar{x}) = m(f;K,\rho)$.*

(c) *If $m(f) = \min f :=$ minimum value of f on K, then* meas $\{x \in K : f(x) = m(f) = \min f\} > 0$.

Proof. The existence of a global minimizer of $\theta(\cdot;K,\rho,f)$ follows from its convexity (and thus continuity) and its coercivity on $\mathbb{R}$. Moreover, such a global minimizer $\bar{m}$ is characterized by the inclusion $0 \in \partial\theta(\bar{m};K,\rho,f)$, with the latter subgradient equal to the interval

$$\left[\,-\int_K \rho'_+(f(x)-\bar{m})dx,\ -\int_K \rho'_-(f(x)-\bar{m})dx\,\right].$$

Thus, except for the uniqueness of the minimizer of $\theta(\cdot;K,\rho,f)$, (a) holds. By way of contradiction, we assume that no $\bar{x} \in \text{int}\ K$ exists satisfying $\bar{m} = f(\bar{x})$. The function $f(x) - \bar{m}$ then never vanishes on int K. Without loss of generality, we may assume that $f(x) - \bar{m}$ is positive on int K. By convexity of ρ, it follows that $\rho'_-(f(x)-\bar{m})$ is positive on int K, which implies

$$\int_K \rho'_-(f(x)-\bar{m})\,dx \;>\; 0$$

because ∂K has measure zero. The above inequality contradicts the variational characterization of $\bar{m}$. It remains to show the uniqueness of $m(f;K,\rho)$. Suppose there are two distinct minimizers m_1 and m_2. Since $\theta(\cdot;K,\rho,f)$ is convex, $\frac{1}{2}m_1 + \frac{1}{2}m_2$ is also a minimizer. Therefore,

$$\int_K \left[\,\rho(f(x)-m_1/2-m_2/2) - \tfrac{1}{2}\,\rho(f(x)-m_1) - \tfrac{1}{2}\,\rho(f(x)-m_2)\,\right]\,dx \;=\; 0.$$

Since the integrand on the left-hand side is continuous and nonpositive, it follows that

$$\rho(f(x)-m_1/2-m_2/2) \;=\; \tfrac{1}{2}\,\rho(f(x)-m_1) + \tfrac{1}{2}\,\rho(f(x)-m_2)$$

for all $x \in K$. By what has been proved above, it follows that there exist $x^1 \neq x^2$ in K such that $f(x^i) = m_i$ for $i = 1, 2$. Hence, by the connectedness of K, there exists $x^* \in K$ such that

$$f(x^*) \;=\; \tfrac{1}{2}\,m_1 + \tfrac{1}{2}\,m_2.$$

Since ρ is a nonnegative function, we deduce $\rho(f(x^*) - m_1) = 0 = \rho(f(x^*) - m_2)$, which yields $f(x^*) = m_1 = m_2$, a contradiction. Finally, to prove (c), assume for contradiction that the measure of the set in question is zero. We then have

$$\int_K \rho'_-(f(x) - \min f)\, dx \,=\, \int_{K_+} \rho'_-(f(x) - \min f)\, dx,$$

where $K_+ := \{x \in K : f(x) > \min f = m(f)\}$. By assumption, meas $K_+ =$ meas $K > 0$. Since $\rho'_-(f(x) - \min f) > 0$ on K_+, it follows that

$$\int_K \rho'_-(f(x) - \min f)\, dx \,>\, 0,$$

which contradicts the characterization of $m(f)$ because $\min f = m(f)$ by assumption. ∎

It is clear that for any constant $c > 0$, $m(f + c) = m(f) + c$. The next result identifies an important monotonicity property of the continuous M-estimator.

THEOREM 2.3. *Let f and g be in $C(K)$ such that $f \geq g$ on K. It holds that $m(f) \geq m(g)$; moreover, strict inequality holds if either* (a) *$f \neq g$, ρ is differentiable, and ρ' is strictly increasing or* (b) *$f > g$ in the interior of K.*

Proof. Assume for contradiction that $m(f) < m(g)$. We have

$$\int_K \rho'_+(g(x) - m(g))\, dx \,\geq\, 0 \,\geq\, \int_K \rho'_-(f(x) - m(f))\, dx.$$

Since ρ'_- is nondecreasing, we deduce

$$\int_K \rho'_-(f(x) - m(f))\, dx \,\geq\, \int_K \rho'_-(g(x) - m(f))\, dx$$

and

$$\int_K \rho'_+(g(x) - m(f))\, dx \,\geq\, \int_K \rho'_+(g(x) - m(g))\, dx.$$

Consequently,

$$\int_K \rho'_+(g(x) - m(f))\, dx \,\geq\, 0 \,\geq\, \int_K \rho'_-(g(x) - m(f))\, dx.$$

By the variational characterization and uniqueness of $m(g)$, it follows that $m(f) = m(g)$, which is a contradiction. If ρ is differentiable, then $m(f)$ and $m(g)$ are the unique scalars m_f and m_g that satisfy the equations

$$\int_K \rho'(f(x) - m(f))\, dx \,=\, 0 \quad \text{and} \quad \int_K \rho'(g(x) - m(g))\, dx \,=\, 0,$$

respectively. If $f \neq g$, then there must exist an open set $\mathcal{O}$ contained in the interior of K such that $f(x) > g(x)$ for all $x \in \mathcal{O}$. If $m(f) = m(g)$, the above characterizations of $m(f)$ and $m(g)$ immediately yield a contradiction, using the fact that $f > g$ on the open set $\mathcal{O} \subset K$. Consequently, strict monotonicity of m (i.e., $m(f) > m(g)$) holds under (a).

Assume (b). Writing $\theta_f := \theta(\cdot; K, \rho, f)$ and $\theta_g := \theta(\cdot; K, \rho, g)$, we divide the remaining proof into six steps.

Step 1. Since $f > g$ in the interior of K, we must have

$$\text{meas}\,\{\, x \in K \,:\, g(x) > m(g)\,\} \le \text{meas}\,\{\, x \in K \,:\, f(x) > m(g)\,\};$$

the claim is that strict inequality also holds:

$$\text{meas}\,\{x \in K : g(x) > m(g)\} < \text{meas}\,\{x \in K : f(x) > m(g)\}. \tag{2.5}$$

To prove this assertion, we distinguish between two cases (i) $m(g) = \min g$ and (ii) $m(g) > \min g$.

Step 2. Assume that $m(g) = \min g$. Since $\{x \in \text{int}\, K : f(x) > m(g)\} = \text{int}\, K$ and meas $\partial K = 0$, it follows that

$$\begin{aligned} &\text{meas}\,\{\, x \in K \,:\, f(x) > m(g)\,\} \\ &\quad = \text{meas}\,\{\, x \in \text{int}\, K \,:\, f(x) > m(g)\,\} = \text{meas int}\, K = \text{meas}\, K \\ &\quad = \text{meas}\,\{\, x \in K \,:\, g(x) = \min g = m(g)\,\} + \text{meas}\,\{\, x \in K \,:\, g(x) > m(g)\,\} \\ &\quad > \text{meas}\,\{\, x \in K \,:\, g(x) > m(g)\,\}, \end{aligned}$$

where the last inequality follows from Theorem 2.2(c). Consequently, (2.5) holds in case (i).

Step 3. In case (ii), we have $\min g < m(g)$. Consider the sets

$$E := \{\, x \in K \,:\, g(x) \ge m(g)\,\} \quad \text{and} \quad U := \{\, x \in K \,:\, f(x) > m(g)\,\}.$$

Since E is contained in the closure of U, we must have meas $U \ge$ meas E. The desired inequality (2.5) follows readily if we can show meas $U >$ meas E. Assume for contradiction that meas $U =$ meas E. It then follows that meas $(U \setminus E) = 0$. But the only way the set $U \setminus E$, which is open in K, has zero measure is when it is empty. This means $U = E$. So $E = U$ is both open and closed in K, which is a connected set. Hence either $E = U = K$ or $E = U = \emptyset$. The former is not possible because $m(g) > \min g$; the latter is not possible because $m(g)$ belongs to the range $g(K)$. This establishes the claim, and thus (2.5) too.

Step 4. Inequality (2.5) implies

$$-\int_K \rho'_{\mp}(f(x) - m(g))\, dx \;<\; -\int_K \rho'_{\mp}(g(x) - m(g))\, dx.$$

Consequently,

$$(\theta_f)'_{\pm}(m(g)) \;<\; (\theta_g)'_{\pm}(m(g)). \tag{2.6}$$

Since $(\theta_f)'_+(m(f)) \ge 0$, to show $m(f) > m(g)$, it suffices to show

$$(\theta_f)'_+(m(g)) \;<\; 0. \tag{2.7}$$

Note that (2.6) is only good enough to imply that $(\theta_f)'_-(m(g)) < 0$, which is weaker than (2.7). If either θ_f or θ_g is differentiable at $m(g)$, then (2.7) follows readily. This is clear in the former case, whereas in the latter case we have $(\theta_f)'_+(m(g)) < (\theta_g)'(m(g)) = 0$.

Step 5. If θ_f and θ_g both fail to be differentiable at $m(g)$, let

$$f_\alpha := \alpha\, f + (1-\alpha)\, g \quad \text{and} \quad \theta_\alpha(m) := \int_K \rho(f_\alpha(x) - m) dx, \quad \alpha \in [0,1].$$

We claim that there exists $\alpha^* \in (0,1)$ such that θ_{α^*} is differentiable at $m(g)$. With this claim established, we can complete the proof as follows. Since $f > f_{\alpha^*} > g$ in the interior of K, together with the result in Step 4 and part (a) proved above, we arrive at $m(f) \geq m(f_{\alpha^*}) > m(g)$. Thus we are left only with the proof of the last claim.

Step 6. Let B be the set of points at which ρ fails to be differentiable. For each $t \in B$ and $\alpha \in (0,1)$, define

$$N_{\alpha,t} := \{ x \in \text{int } K : f_\alpha(x) - m_g = t \}.$$

Note that for fixed $t \in B$ and $\alpha > \alpha'$, the sets $N_{\alpha,t}$ and $N_{\alpha',t}$ are disjoint. Since K has finite measure, for each $t \in B$ and each $n = 1, 2, \ldots$, only a finite number of α satisfies meas $N_{\alpha,t} > 1/n$. Hence the set $\{\alpha \in (0,1) : \text{meas } N_{\alpha,t} > 0\}$ is countable. Since B is countable, it follows that

$$\{\alpha \in (0,1) : \text{meas } N_{\alpha,t} > 0 \text{ for some } t \in B\} = \bigcup_{t \in B} \{\alpha \in (0,1) : \text{meas } N_{\alpha,t} > 0\}$$

is countable. But $(0,1)$ is uncountable, so there exists $\alpha^* \in (0,1)$ such that

$$\begin{aligned} 0 = \text{meas} \bigcup_{t \in B} N_{\alpha^*,t} &= \text{meas } \{ x \in \text{int } K : f_{\alpha^*}(x) \in B \} \\ &= \text{meas } \{ x \in K : f_{\alpha^*}(x) \in B \}. \end{aligned}$$

By part (f) of Proposition 2.1, it follows that θ_{α^*} is differentiable at $m(g)$. □

Remark. It is in general not true that if $f \geq g$ and $f \neq g$, then $m(f) > m(g)$, even if f is almost everywhere strictly greater than g. An example is if f and g are continuous functions defined on an interval I and are both strictly increasing, and $f(x) > g(x)$ except at the midpoint of I. With $\rho = |\cdot|$, it follows that $m(f) = \text{med}(f|I) = f(\text{midpoint of } I) = g(\text{midpoint of } I) = \text{med}(g|I) = m(g)$. □

Part (a) of the next result implies that the continuous M-estimator $m(f)$ is nonexpansive, and thus continuous, in its argument; the second part extends part (b) of Theorem 2.2.

COROLLARY 2.4. *The following two statements hold.*

(a) *The continuous M-estimator m is nonexpansive on $C(K)$; i.e.,*

$$| m(f) - m(g) | \leq \max_{x \in K} | f(x) - g(x) | \quad \forall f, g \in C(K).$$

(b) *If f and g in $C(K)$ are such that $m(f) = m(g)$, then $x \in \text{int } K$ exists such that $f(x) = g(x)$.*

Proof. Let $\sigma := \max_{x \in K} |f(x) - g(x)|$. We have $g - \sigma \leq f \leq g + \sigma$ on K. An application of the monotonicity of m yields

$$m(g) - \sigma = m(g - \sigma) \leq m(f) \leq m(g + \sigma) = m(g) + \sigma,$$

from which the desired nonexpansiveness of m follows readily.

To prove statement (b), assume for contradiction that $f(x) \neq g(x)$ for all $x \in \text{int } K$. It then follows that either $f > g$ on int K or $f < g$ on int K. Either case yields a contradiction to the assumption that $m(f) = m(g)$ by the strict monotonicity of m. □

The next result asserts a "strong minimizing" property of $m(f; K, \rho)$ associated with a C^2 function ρ with positive second derivatives. A consequence of this property is that a global a posteriori error bound exists for the continuous M-estimator; see the last inequality in the proposition below.

PROPOSITION 2.5. *Let $f \in C(K)$ and $\rho \in C^2$ be given. If $\rho''(f(x) - m(f))$ does not vanish identically on K, then $\theta''(m(f); K, f, \rho) > 0$. Hence, positive constants c and δ exist such that*

$$(2.8) \qquad |\, m - m(f)\,| \le \delta \;\Rightarrow\; \theta(m; K, f, \rho) - \theta(m(f); K, f, \rho) \ge c\,|\, m - m(f)\,|^2$$

and

$$(2.9) \qquad |\, m - m(f)\,| > \delta \;\Rightarrow\; \theta(m; K, f, \rho) - \theta(m(f); K, f, \rho) \ge c\,\delta\,|\, m - m(f)\,|.$$

Consequently, there exists $\eta > 0$ such that, for all $m \in \mathbb{R}$,

$$\begin{aligned} &|\, m - m(f)\,| \\ &\le \eta \max\left\{ \theta(m; K, f, \rho) - \theta(m(f); K, f, \rho),\ \sqrt{\theta(m; K, f, \rho) - \theta(m(f); K, f, \rho)} \right\}. \end{aligned}$$

Proof. Write $\theta := \theta(\cdot; K, f, \rho)$. By part (d) of Proposition 2.1, θ'' exists. Moreover, the expression (2.4) shows that θ'' is nonnegative on $\mathbb{R}$. If $\theta''(m(f)) = 0$, then since f is continuous on K and ρ'' is nonnegative and continuous on $\mathbb{R}$, it follows that $\rho''(f(x) - m(f)) = 0$ for all $x \in K$. But this contradicts the assumption that the latter function does not vanish identically on K. Consequently, $\theta''(m(f)) > 0$.

The existence of δ and c satisfying (2.8) is a standard second-order consequence of the minimizing property of $m(f)$ and of the positivity of $\theta''(m(f); K, f, \rho)$. To prove (2.9), let m be such that $m - m(f) > \delta$. (The proof of the case $m - m(f) < -\delta$ is similar and omitted.) Define $m' := m(f) + \delta$. We then have

$$m' = \frac{\delta}{m - m(f)}\, m + \frac{m - m(f) - \delta}{m - m(f)}\, m(f),$$

which implies, by convexity of θ,

$$\theta(m') \le \frac{\delta}{m - m(f)}\, \theta(m) + \frac{m - m(f) - \delta}{m - m(f)}\, \theta(m(f)).$$

Since $\theta(m') \ge \theta(m(f)) + c(m' - m(f))^2 = \theta(m(f)) + c\delta^2$, (2.9) follows readily. The last assertion of the proposition is immediate from (2.8) and (2.9). □

3. Interpolation of continuous M-estimators. In this and the next section, we consider the interpolation of the continuous M-estimators of polynomials on given intervals. This section establishes the well-posedness of the interpolation problem using nonconstructive topological arguments, generalizing the main result in [6]. The next section discusses Newton's method for solving the interpolation problem with a smooth ρ.

Denote by Π_n the space of all polynomials of degree $\le n$. Let I be a given closed finite interval. For any nonzero $p \in \Pi_n$ and any $m \in \mathbb{R}$, the set $p^{-1}(m) \cap I$ has at most n elements. Hence, the set

$$\{t \in I : \rho \text{ is not differentiable at } p(t) - m\}$$

is countable; consequently, by part (f) of Proposition 2.1, the function $\theta(m)$ is differentiable everywhere on $\mathbb{R}$. Hence, for $p \in \Pi_n$ we have

$$(3.1) \qquad \int_I \rho'_{\pm}(p(t) - m(p\,; I, \rho))\, dt = 0.$$

We introduce the map $\mathbf{M}_\rho : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ associated with $n+1$ *nonoverlapping* compact intervals I_i, for $i = 0, \ldots, n$, each with a nonempty interior, where nonoverlapping means $\operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset$ for all $i \neq j$. For each vector $a \in \mathbb{R}^{n+1}$ with components a_i for $i = 0, \ldots, n$, let $\mathbf{M}_\rho(a)$ be the $(n+1)$-vector whose ith component, for $i = 0, \ldots, n$, is equal to $m(p\,; I_i, \rho)$, where p is the polynomial in Π_n with coefficients a_j; i.e.,

$$p(t) \equiv \sum_{j=0}^{n} a_j\, t^j, \quad t \in \mathbb{R}. \tag{3.2}$$

The goal of this section is to establish the following main result, which immediately implies the *well-posedness* of the interpolation problem of continuous M-estimators: given any values m_i, $i = 0, \ldots, n$, there exists a unique $p \in \Pi_n$ such that $m(p\,; I_i, \rho) = m_i$ for all i; moreover, such an interpolant p depends continuously on the data m_i.

THEOREM 3.1. *$\mathbf{M}_\rho$ is a homeomorphism from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^{n+1}$.*

Proof. It suffices to show that $\mathbf{M}_\rho$ is continuous, injective, and norm-coercive [8, Theorem 5.3.8]. By Corollary 2.4(a), $\mathbf{M}_\rho$ is Lipschitz continuous. Indeed, for any two vectors a and b in $\mathbb{R}^{n+1}$ with associated polynomials p and q, we have, for every $i = 0, 1, \ldots, n$,

$$|\, m(p\,; I_i, \rho) - m(q\,; I_i, \rho)\,| \,\leq\, \sup_{t \in I_i} |\, p(t) - q(t)\,| \,\leq\, L_i \,\|\, a - b\,\|,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^{n+1}$ and $L_i > 0$ is a constant that depends on the interval I_i and is independent of ρ and the vectors a and b. Letting $L = \max_{0 \leq i \leq n} L_i$, we deduce

$$\|\, \mathbf{M}_\rho(a) - \mathbf{M}_\rho(b)\,\| \,\leq\, L \,\|\, a - b\,\| \quad \forall\, \rho \text{ and } \quad \forall\, a, b \,\in\, \mathbb{R}^{n+1}. \tag{3.3}$$

Note that the Lipschitz constant L is independent of ρ; hence the family of maps $\{\mathbf{M}_\rho : \rho\}$ is *equi-Lipschitz continuous*. Injectivity of $\mathbf{M}_\rho$ follows easily from Corollary 2.4(b): If $p, q \in \Pi_n$ are such that $\mathbf{M}_\rho(p) = \mathbf{M}_\rho(q)$, then there exists, for every $i = 0, \ldots, n$, a point $t_i \in \operatorname{int} I_i$ such that $p(t_i) = q(t_i)$; this implies $p = q$. It remains to show norm-coerciveness; i.e., we need to show that

$$\lim_{\|a\| \to \infty} \|\, \mathbf{M}_\rho(a)\,\| \,=\, \infty. \tag{3.4}$$

We divide the remaining proof into several major steps.

Step 1. Define, for any scalar $c > 0$, the function $\rho_c : \mathbb{R} \to \mathbb{R}$ by $\rho_c(t) := \rho(ct)$ for all $t \in \mathbb{R}$. Observe that for any interval I and any function $f \in C(I)$, we have

$$\begin{aligned} m(f; I, \rho) \,&=\, \arg\min_m \int_I \rho(f(t) - m)\, dt \,=\, \arg\min_m \int_I \rho_c \left(\frac{f(t)}{c} - \frac{m}{c} \right) dt \\ &=\, c\, m \left(\frac{f}{c}; I, \rho_c \right). \end{aligned}$$

It follows that for a nonzero vector $a \in \mathbb{R}^{n+1}$,

$$\mathbf{M}_\rho(a) \,=\, \|\, a\,\|\, \mathbf{M}_{\rho_{\|a\|}}\, (a/\|a\|)\,. \tag{3.5}$$

Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. Clearly, if

$$\inf \{\, \|\, \mathbf{M}_{\rho_c}(a)\,\| \,:\, c \,\geq\, 1,\; a \,\in\, S^n \,\} \,>0, \tag{3.6}$$

then (3.4) follows. It is clear that (3.6) implies

$$\inf\{\,\|\,\mathbf{M}_{\rho_c}(a)\,\|\;:\;c\,\geq\,1\,\}\,>0\quad\forall\,a\,\in\,S^n. \tag{3.7}$$

Thanks to the compactness of S^n and the equi-Lipschitz continuity of $\{\mathbf{M}_{\rho_c} : c > 0\}$, the converse implication also holds. Indeed, assume that (3.7) holds but (3.6) does not. There exist sequences $\{a^k\} \subset S^n$ and $\{c_k\} \in [1, \infty)$ such that $\mathbf{M}_{\rho_{c_k}}(a^k) \to 0$. Since S^n is compact, we may assume without loss of generality that the sequence $\{a^k\}$ converges to some $a^\infty \in S^n$. By (3.3), we have

$$\|\,\mathbf{M}_{\rho_{c_k}}(a^k) - \mathbf{M}_{\rho_{c_k}}(a^\infty)\,\|\;\leq\;L\,\|\,a^k - a^\infty\,\|\quad\forall\,k.$$

Consequently, it follows that

$$\lim_{k\to\infty}\mathbf{M}_{\rho_{c_k}}(a^\infty)\;=\;0,$$

which contradicts (3.7). Hence (3.6) $\Longleftrightarrow$ (3.7).

Step 2. Let $a \in \mathbb{R}^{n+1}$ be an arbitrary nonzero vector, and let $p \in \Pi_n$ be given by (3.2). Since p has no more than n roots and since there are $(n+1)$ nonoverlapping intervals I_i, it follows that one of two cases must hold:

(i) there exists $j \in \{0, \dots, n\}$ such that $p > 0$ or $p < 0$ on I_j, or

(ii) $n \geq 1$ and there exists $j \in \{0, \dots, n-1\}$ such that I_j and I_{j+1} intersect at a common endpoint where p vanishes and $p(x)p(y) < 0$ for all $x \in \operatorname{int} I_j$ and $y \in \operatorname{int} I_{j+1}$.

To see this, note the following special property of a polynomial. Namely, if a polynomial does not change sign and does not vanish in an interval except at one interior point of the interval, then the latter point must be a root of the polynomial of multiplicity at least two. By this property and a straightforward counting argument, one can easily show that if neither of the above two cases hold, then, counting multiplicities, p must have at least $n+1$ zeros, which is impossible. To complete the proof of the theorem, we show that (3.7) holds in either case (i) or (ii) above.

Step 3. In case (i) there exists $\delta > 0$ such that p or $-p \geq \delta$ on I_j. Since $m(p\,; I_j, \rho_c) = p(t_j)$ for some $t_j \in I_j$, it follows that $\|\mathbf{M}_{\rho_c}(a)\| \geq \delta$ for all $c > 0$.

Step 4. In case (ii) we prove (3.7) by contradiction. If (3.7) fails, then there exists a sequence $\{c_k\}$ in $[1, \infty)$ such that

$$\lim_{k\to\infty} m(p\,; I_i, \rho_{c_k})\;=\;0\quad\text{ for } i = j \text{ and } j+1.$$

Without loss of generality, we may assume that

$$p\;<\;0 \text{ on } \operatorname{int} I_j\quad\text{ and }\quad p\;>\;0 \text{ on } \operatorname{int} I_{j+1}.$$

For $i = j,\, j+1$, write $m_{k,i} := m(p\,; I_i, \rho_{c_k})$ and $\theta_{k,i} := \theta(\cdot; I_i, p, \rho_{c_k})$. By part (c) of Theorem 2.2, it follows that $m_{k,j} < 0$ and $m_{k,j+1} > 0$ for all k.

By (3.1), we have the following optimality condition:

(3.8)
$$\int_{I_j}\rho'_\pm\,(c_k[p(t) - m_{k,j}])\;dt\;=\;0\;=\;\int_{I_{j+1}}\rho'_\pm\,(c_k[p(t) - m_{k,j+1}])\;dt\quad\forall\,k = 1, 2, \dots\,.$$

Since $\operatorname{meas}\{t \in I_i : p(t) = 0\} = 0$, an elementary property of measure gives

$$\lim_{m\uparrow 0}\,\operatorname{meas}\,\{t \in I_j : p(t) \geq m\} = 0 = \lim_{m\downarrow 0}\,\operatorname{meas}\,\{t \in I_{j+1} : p(t) \leq m\}. \tag{3.9}$$

By continuity of p, there exist $\varepsilon > 0$, $\delta > 0$, $J_i \subset I_i$ with meas $J_i \geq \varepsilon$ for $i = j, j+1$ such that $p \leq -\delta$ and $p \geq \delta$ on J_j and J_{j+1}, respectively. Pick k large enough such that $m_{k,j+1} - m_{k,j} < \delta$ (in particular, $-\delta < m_{k,j} < 0$ and $0 < m_{k,j+1} < \delta$),

$$\text{meas}\,\{t \in I_j : p(t) > m_{k,j}\} < \varepsilon \quad \text{and} \quad \text{meas}\,\{t \in I_{j+1} : p(t) < m_{k,j+1}\} < \varepsilon.$$

We then have

$$0 = \int_{I_j} (\rho_{c_k})'_+(p(t) - m_{k,j})\,dt = T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 &:= \int_{\{t:p(t)\leq -\delta\}} (\rho_{c_k})'_+(p(t) - m_{k,j})\,dt \\ &\leq \int_{\{t:p(t)\leq -\delta\}} (\rho_{c_k})'_+(-\delta - m_{k,j})\,dt \leq \varepsilon(\rho_{c_k})'_+(-m_{k,j+1}) \\ T_2 &:= \int_{\{t:-\delta<p(t)<m_{k,j}\}} (\rho_{c_k})'_+(p(t) - m_{k,j})\,dt < 0 \\ T_3 &:= \int_{\{t:p(t)\geq m_{k,j}\}} (\rho_{c_k})'_+(p(t) - m_{k,j})\,dt \\ &\leq \int_{\{t:p(t)>m_{k,j}\}} (\rho_{c_k})'_+(-m_{k,j})\,dt \leq \varepsilon\,(\rho_{c_k})'_+(-m_{k,j}), \end{aligned}$$

by the monotonicity of $(\rho_{c_k})'_+$. Hence

$$0 < (\rho_{c_k})'_+(-m_{k,j+1}) + (\rho_{c_k})'_+(-m_{k,j}).$$

Similarly, using

$$0 = \int_{I_{j+1}} (\rho_{c_k})'_+(p(t) - m_{k,j+1})\,dt,$$

we can deduce $0 > (\rho_{c_k})'_+(-m_{k,j}) + (\rho_{c_k})'_+(-m_{k,j+1})$, which is a contradiction. □

4. Newton's method for interpolation. By Theorem 3.1, the equation

$$\mathbf{M}_\rho(a) = b \tag{4.1}$$

has a unique solution for every vector $b \in \mathbb{R}^{n+1}$; moreover, such a solution depends continuously on b. A natural approach to solve (4.1) is by Newton's method, possibly with a globalization scheme [2]. In order for this method to be directly applicable, we take ρ to be twice continuously differentiable throughout this section. This smoothness assumption allows us to completely bypass the difficulty in dealing with the nonsmoothness of the original median-interpolation problem considered in [3, 6]. Moreover, by the theorem below, if $\rho'' > 0$, then $\mathbf{M}_\rho$ is a diffeomorphism on $\mathbb{R}^{n+1}$; i.e., both $\mathbf{M}_\rho$ and $(\mathbf{M}_\rho)^{-1}$ are differentiable maps. In particular, $\mathbf{M}_\rho$ has a nonsingular Jacobian matrix everywhere on $\mathbb{R}^{n+1}$.

THEOREM 4.1. *Let I_i, $i = 0, \ldots, n$, be $n+1$ nonoverlapping compact intervals, each with a nonempty interior. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable* *function with* $\rho'' > 0$ *on* $\mathbb{R}$. *If* ρ *has a minimizer at zero such that* $\rho(0) = 0$, *then* $\mathbf{M}_\rho$ *is a diffeomorphism from* $\mathbb{R}^{n+1}$ *onto itself.*

Proof. For simplicity, we write $\mathbf{M}$ for $\mathbf{M}_\rho$. The positivity of ρ'' implies that ρ is strictly convex on $\mathbb{R}$. Hence, zero is the unique minimizer of ρ. For $i = 0, \dots, n$, write

$$\theta_i(m, a) := \int_{I_i} \rho \left(\sum_{j=0}^{n} a_j\, t^j - m \right) dt. \tag{4.2}$$

As a function of the two arguments $(m, a) \in \mathbb{R}^{1+(n+1)}$, θ_i is twice continuously differentiable. For any fixed but arbitrary vector $a \in \mathbb{R}^{n+1}$, the continuous M-estimator $\mathbf{M}_i(a)$ is the unique root m of the equation

$$\frac{\partial \theta_i(m, a)}{\partial m} = 0,$$

which is a parametric, nonlinear equation with m as the primary variable and a as the parameter. By Proposition 2.5,

$$\frac{\partial^2 \theta_i(\mathbf{M}_i(a), a)}{\partial m^2} > 0.$$

Therefore, by the implicit-function theorem, it follows that $\mathbf{M}_i$ (and thus $\mathbf{M}$) is a differentiable function on $\mathbb{R}^{n+1}$. We claim that the Jacobian matrix of $\mathbf{M}$ at $a \in \mathbb{R}^{n+1}$, denoted $J\mathbf{M}(a)$, is nonsingular. Note that

$$\frac{\partial \mathbf{M}_i(a)}{\partial a_j} = \left(\frac{\partial^2 \theta_i(\mathbf{M}_i(a), a)}{\partial m^2} \right)^{-1} \left(\frac{\partial^2 \theta_i(\mathbf{M}_i(a), a)}{\partial m\, \partial a_j} \right). \tag{4.3}$$

If $e \in \mathbb{R}^{n+1}$ is such that $J\mathbf{M}(a)e = 0$, then we have

$$\sum_{j=0}^{n} \frac{\partial^2 \theta_i(\mathbf{M}_i(a), a)}{\partial m\, \partial a_j}\, e_j = 0 \quad \forall\, i = 0, \dots, n.$$

By a direct differentiation, we can see that the above is equivalent to

$$\int_{I_i} \rho'' \left(\sum_{\ell=0}^{n} a_\ell\, t^\ell - \mathbf{M}_i(a) \right) q(t)\, dt = 0, \qquad i = 0, \dots, n,$$

where $q(t) \equiv \sum_{j=0}^{n} e_j\, t^j$. Since ρ'' is everywhere positive, it follows that q has at least one zero in int I_i for every $i = 0, \dots, n$. Consequently, the polynomial $q(t)$, which is of degree n, has $n + 1$ distinct real roots. This is not possible unless $q(t)$ is the zero polynomial. Hence $J\mathbf{M}(a)$ must be nonsingular. This implies that $\mathbf{M}^{-1}$ is differentiable; hence $\mathbf{M}$ is a diffeomorphism on $\mathbb{R}^{n+1}$. ☐

The condition $\rho'' > 0$ cannot be dispensed with in the above theorem. For example consider $\rho = |\cdot|^p$; then $\rho \in \mathrm{C}^2$ when $p \geq 2$. On the other hand, $\mathbf{M}_{|\cdot|^p}$ is homogeneous of degree 1; i.e.,

$$\mathbf{M}_{|\cdot|^p}(c\, a) = c\, \mathbf{M}_{|\cdot|^p}(a); \;\; \text{thus} \;\; D_v \mathbf{M}_{|\cdot|^p}(0) = \mathbf{M}_{|\cdot|^p}(v),$$

where D_v denotes the directional derivative operator in the direction v. This implies that $\mathbf{M}_{|\cdot|^p}$ is differentiable at the origin if and only if $\mathbf{M}_{|\cdot|^p}$ is a linear map; but an elementary calculation shows that (unless $n \leq 1$) the latter is true when and only when $p = 2$.

4.1. Method I. With the above preparation, we can formally state a globally convergent Newton method for solving (4.1); see [2, 11]. In essence, this is the damped Newton method that involves two main computational steps in each iteration: the first step is solving the Newton equation

$$\mathbf{M}_\rho(a^k) + J\mathbf{M}_\rho(a^k)\, da^k \;=\; b \tag{4.4}$$

for the direction da^k at the current iterate a^k; the second step is performing an Armijo line search [1, section 1.2, p. 29] on the merit function

$$\phi(a) \;:=\; \tfrac{1}{2}\,(\,\mathbf{M}_\rho(a) - b\,)^T(\,\mathbf{M}_\rho(a) - b\,)$$

at the current iterate along the computed Newton direction. Normally, given scalars σ' and β in $(0,1)$, the latter line search calls for the determination of the smallest nonnegative integer i satisfying

$$\phi(a^k + \beta^i da^k) - \phi(a^k) \;\le\; \sigma'\,\beta^i\,\nabla\phi(a^k)^T da^k.$$

With the function ϕ on hand and the search direction da^k computed from (4.4), we have

$$\nabla\phi(a^k) \;=\; J\mathbf{M}_\rho(a^k)^T(\,\mathbf{M}_\rho(a^k) - b^k\,),$$

which yields $\nabla\phi(a^k)^T da^k \;=\; -2\phi(a^k)$. Noting the latter identity, we formulate the following algorithm.

Newton's Algorithm for Interpolation.

(a) (inputs). Let the function ρ and the $(n+1)$ intervals I_i satisfy the assumptions in Theorem 4.1. Let $b \in \mathbb{R}^{n+1}$ and $a^0 \in \mathbb{R}^{n+1}$ be given. Let σ and β be given scalars in $(0,1)$. Set $k = 0$.

(b) (solving linear equations). Solve (4.4) for the Newton direction da^k.

(c) (Armijo line search). Let i_k be the smallest nonnegative integer i such that

$$\phi(a^k + \beta^i da^k) \;\le\; (\,1 - \sigma\,\beta^i\,)\,\phi(a^k).$$

Set $\tau_k := \beta^{i_k}$ and $a^{k+1} := a^k + \tau_k da^k$. Let $k \leftarrow k+1$.

(d) If $\|\mathbf{M}_\rho(a^k) - b\| \le$ tolerance, stop. Otherwise return to step (b). □

While the convergence of Newton's method is well known (cf. [2, Chapter 6] or [5, Chapter 8], e.g.), for completeness we give a sketch of the proof of the following result, omitting some details.

Theorem 4.2. *Under the assumptions of Theorem* 4.1*, the algorithm as described above generates a well-defined sequence* $\{a^k\}$ *that converges Q-superlinearly to the unique solution of* (4.1)*. Moreover,* $i_k = 0$ *for all but finitely many* k*. Finally, if* ρ'' *is Lipschitz continuous on* $\mathbb{R}$*, then the convergence is Q-quadratic.*

Proof. The well-definedness of each search direction da^k is ensured by the nonsingularity of $J\mathbf{M}_\rho(a)$ on $\mathbb{R}^{n+1}$. Since the method is essentially a gradient-related line-search method applied to the unconstrained minimization of the merit function $\phi(a)$, standard results from nonlinear programming, such as [1, Proposition 1.2.1], guarantee that the sequence $\{a^k\}$ is well defined and that every accumulation point of the sequence is a stationary point of ϕ. At least one such point must exist because ϕ has bounded level sets; in turn, the latter is due to the diffeomorphism property of $\mathbf{M}_\rho$. Since $\nabla\phi(a) = J\mathbf{M}_\rho(a)^T(\mathbf{M}_\rho(a) - b)$ and $J\mathbf{M}_\rho(a)$ is a nonsingular matrix for all a, it follows that every accumulation point of the sequence $\{a^k\}$ satisfies (4.1).

Since the latter equation has a unique solution, it follows that $\{a^k\}$ converges to the unique solution of (4.1). Letting a^* be the unique zero of (4.1), we note that $\nabla^2\phi(a^*) = J\mathbf{M}_\rho(a^*)^T J\mathbf{M}_\rho(a^*)$ is positive definite. Therefore, by verifying the well-known Dennis–Moré condition (see [2, equation (6.3.10)]), Theorem 6.3.4 in the latter reference immediately yields the assertion about the ultimate attainment of a unit step size (i.e., $i_k = 0$ for all but finitely many k). This in turn readily implies the quadratic convergence statement. ∎

4.2. Method II. We present below an alternative formulation of the system of equations (4.1) which results in a total bypass of the evaluation of $\mathbf{M}_\rho(a)$. Nevertheless, we should caution the reader that there is a theoretical difference between the alternative formulation and the original equation (4.1) that will become clear in the following discussion.

Since $\mathbf{M}_i(a) \equiv (\mathbf{M}_\rho)_i(a)$ satisfies $\int_{I_i} \rho'(p(t) - \mathbf{M}_i(a))\,dt = 0$, it is clear that (4.1) is equivalent to

$$\int_{I_i} \rho' \left(\sum_{j=0}^{n} a_j t^j - b_i \right) dt = 0, \quad i = 0, \dots, n. \tag{4.5}$$

Define the C^1 map $F_\rho := F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, where $F_i(a)$ is the left-hand side of the above equation. It is useful to clarify the difference between the two functions $\mathbf{M}$ and F in terms of the following function:

$$\psi_i(m, a) := \int_{I_i} \rho' \left(\sum_{j=0}^{n} a_j t^j - m \right) dt = -\frac{\partial \theta_i(m, a)}{\partial m}, \quad (m, a) \in \mathbb{R}^{1+(n+1)},$$

where $\theta_i(m, a)$ is given by (4.2). While $\mathbf{M}_i(a)$ is the (unique) zero of $\psi_i(\cdot, a)$ on $\mathbb{R}$, $F_i(a)$ is the value of $\psi_i(\cdot, a)$ at a given value $b_i \in \mathbb{R}$.

A similar Newton method can be applied to the equation

$$F(a) = 0. \tag{4.6}$$

It is easy to show that

$$\frac{\partial F_i(a)}{\partial a_j} = \left(\frac{\partial^2 \theta_i(b_i, a)}{\partial m\, \partial a_j} \right). \tag{4.7}$$

Under the assumptions of Theorem 4.1, we can show that $JF(a)$ is nonsingular for all $a \in \mathbb{R}^{n+1}$. Thus F is a local homeomorphism everywhere on $\mathbb{R}^{n+1}$. However, unlike $\mathbf{M}$, F is in general *not* a global homeomorphism. In fact, for a function ρ whose derivative ρ' is bounded (e.g., $\rho(t) \equiv \sqrt{t^2 + c} - \sqrt{c}$ for any $c > 0$), it is clear that F has a bounded range and therefore cannot be norm-coercive, and thus it cannot be a global homeomorphism.

In spite of the theoretical difference, Newton's method can be applied to (4.6). Omitting the details, we summarize the two main computational steps in each iteration. At the beginning of each iteration, an iterate $a^k \in \mathbb{R}^{n+1}$ is given. We then solve for da^k in the equation

$$F(a^k) + JF(a^k)\, da^k = 0$$

and next perform an Armijo line search on the merit function

$$\varphi(a) := \tfrac{1}{2} F(a)^T F(a)$$

starting at a^k and moving along the direction da^k. In the resulting scheme, there is no longer a need to evaluate $\mathbf{M}(a^k)$. The convergence of this alternative application of Newton's method is summarized below.

THEOREM 4.3. *Under the assumptions of Theorem* 4.1*, the algorithm as described above generates a well-defined sequence* $\{a^k\}$*. If the level set*

$$L(a^0) := \{ a \in \mathbb{R}^{n+1} : \| F(a) \|_2 \le \| F(a^0) \|_2 \}$$

is bounded, then $\{a^k\}$ *is bounded, and all other conclusions of Theorem* 4.2 *remain valid.*

Proof. Since the sequence $\{a^k\}$ is contained in the level set $L(a^0)$, the boundedness of $\{a^k\}$ follows from that of the set. The rest of the proof is similar to that of Theorem 4.2. ▯

In summary, when Newton's method is applied to the two equivalent equations, (4.1) and (4.6), the resulting algorithms differ computationally and theoretically. The computational difference lies in the need to evaluate the continuous M-estimators $\mathbf{M}(a)$ at intermediate polynomials. The theoretical difference lies in the choice of initial iterate a^0. With (4.1), there is no restriction on a^0; with (4.6), a^0 should be such that the level set $L(a^0)$ is bounded.

4.3. Smoothed continuous medians. In addition to producing M-estimators that are of independent interest, the choice of a C^2 function ρ can be used to approximate the continuous median. One such family of "smoothed continuous medians" is obtained by letting

$$\rho_\varepsilon(t) := \sqrt{t^2 + \varepsilon^2} - \varepsilon, \quad t \in \mathbb{R},$$

where ε is a positive scalar presumed to be small. Note that ρ_ε is globally Lipschitz continuous on $\mathbb{R}$ and ρ_ε'' is positive everywhere. Notice that

$$0 \le |t| - \rho_\varepsilon(t) < \varepsilon \quad \forall t \in \mathbb{R}.$$

By the next proposition (which is stated in a general context), the above inequality implies that $\{m(f; I_i, \rho_\varepsilon)\}$ converges to $\mathrm{med}(f|I_i)$, as $\varepsilon \downarrow 0$, for all $f \in C(I_i)$. This limit justifies the use of $m(f; I_i, \rho_\varepsilon)$ as an approximation of the continuous median $\mathrm{med}(f|I_i)$.

PROPOSITION 4.4. *Let* $\rho : \mathbb{R} \to \mathbb{R}_+$ *be a convex function with a unique minimizer at zero such that* $\rho(0) = 0$*. For every* $\varepsilon > 0$*, let* ρ_ε *be a convex function with the same properties as* ρ*. Assume that there exists a constant* $\eta > 0$ *such that, for every* $\varepsilon > 0$ *sufficiently small,*

$$|\rho(t) - \rho_\varepsilon(t)| \le \eta\,\varepsilon \quad \forall t \in \mathbb{R};$$

then, for every $f \in C(K)$*,*

$$\lim_{\varepsilon \downarrow 0} m(f; K, \rho_\varepsilon) = m(f; K, \rho).$$

Proof. For every $\varepsilon > 0$, there exists $x(\varepsilon) \in \mathrm{int}\, K$ such that $m(f; K, \rho_\varepsilon) = f(x(\varepsilon))$. Consequently, it follows that

$$\sup_{\varepsilon > 0} |m(f; K, \rho_\varepsilon)| < \infty.$$

To establish the desired limit, it suffices to show that for every sequence of positive scalars $\{\varepsilon_k\}$ converging to zero, the sequence $\{m(f;K,\rho_{\varepsilon_k})\}$ converges to $m(f;K,\rho)$. In turn, it suffices to show that if

$$\lim_{k(\in\kappa)\to\infty} m(f;K,\rho_{\varepsilon_k}) \;=\; m_\infty,$$

where κ is an infinite subset of $\{1,2,\dots\}$, then

$$\int_K \rho(f(x)-m)\,dx \;\geq\; \int_K \rho(f(x)-m_\infty)\,dx \tag{4.8}$$

for all $m \in \mathbb{R}$. Writing $m_k := m(f;K,\rho_{\varepsilon_k})$, we have, for every k,

$$\int_K \rho_{\varepsilon_k}(f(x)-m)\,dx \;\geq\; \int_K \rho_{\varepsilon_k}(f(x)-m_k)\,dx.$$

For all k sufficiently large, we have

$$\int_K \rho_{\varepsilon_k}(f(x)-m)\,dx \;\leq\; \int_K \rho(f(x)-m)\,dx + \eta\,\varepsilon_k\,\text{meas}\,(K)$$

and

$$\int_K \rho_{\varepsilon_k}(f(x)-m_k)\,dx \;\geq\; \int_K \rho(f(x)-m_k)\,dx - \eta\,\varepsilon_k\,\text{meas}\,(K).$$

Passing to the limit $k(\in\kappa)\to\infty$ readily yields the desired inequality (4.8). ◻

4.4. Method II applied to interpolation of medians and smoothed medians. We have implemented a MATLAB solver `MEstimatorInterp` that applies Method II in section 4.2 to the M-estimator interpolation problem. We shall report and compare some of the numerical results based on this solver in the next subsection. In the implementation, we evaluate a polynomial using its Lagrange form. Specifically, for a polynomial L_i in Π_n, we write, relative to the interval I_i,

$$L_i(x) \;:=\; \left[\prod_{j=0,j\neq i}^{n}(\,x-x_i\,)\right]\left[\prod_{j=0,j\neq i}^{n}(\,x_j-x_i\,)\right]^{-1}, \quad x_j \;=\; \text{midpoint of } I_j.$$

The numerical result pertains to the following:

(i) `MEstimatorInterp` applied to the smoothed median-interpolation problem. In this case

$$F_i(a) \;=\; \int_{I_i}\rho'\left(\sum_{i=0}^{n} a_i L_i(t) - b_i\right)\,dt, \quad \rho'(t) = \frac{t}{\sqrt{t^2+0.01}}. \tag{4.9}$$

(ii) `MEstimatorInterp` applied to the median-interpolation problem. In this case

$$F_i(a) \;=\; \int_{I_i}\text{sign}\left(\sum_{i=0}^{n} a_i L_i(t) - b_i\right)\,dt. \tag{4.10}$$

Since $\rho = |\cdot|$ does not satisfy assumptions in Theorem 4.3, there is no guarantee that the method would converge.

(iii) MedianInterp in [4, section 2.1.4] applied to the median-interpolation problem. Based on a fixed-point iteration, an implementation of this solver, whose convergence has not been established, is freely available in WaveLab 802 at http://www-stat.stanford.edu/~wavelab/.

(iv) Same as (ii), except that the initial guess is chosen to be the fourth iterate computed from the fixed-point method in (iii).

Here are some implementation details:

Initial guess. Except in case (iv), we use the midpoint interpolant of the data $p_0 := \sum_{i=0}^{n} b_i L_i$ as the initial guess. This is motivated by the heuristic

$$\mathrm{med}(p|I) \;\approx\; p(\text{midpoint of } I) \;\approx\; m(p; I, \sqrt{(\cdot)^2+\varepsilon^2}-\varepsilon).$$

Computing $F(a)$. The integrals (4.9) and (4.10) are computed using `quad()` in Matlab 6.1. This routine is based on a recursive adaptive Simpson quadrature. The tolerance is set to `tol` $= 10^{-12}$. The evaluations of $\sum_i a_i L_i(t)$ in the integrands of (4.9) and (4.10) are performed using Neville's algorithm.

Computing $JF(a)$. For the computation of the Jacobian of F in the case of (4.9), one can in principle use (4.7), which basically involves an integral with ρ'' appearing in the integrand. This approach, however, turns out to be quite problematic for the purpose of smoothed median interpolation, since in this application $\rho'(x) \approx \mathrm{sign}(x)$ and $\rho''(x) \approx$ the dirac function. Thus we approximate $\partial F_i(a)/\partial a_j$ based on a central divided difference:

$$(4.11)\qquad \frac{\partial F_i(a)}{\partial a_j} \approx \frac{F_i(a+he_j)-F_i(a-he_j)}{2h}, \quad h = 10^{-8} \approx \sqrt{\text{machine precision}},$$

where e_j is the jth vector in the standard basis of $\mathbb{R}^{n+1}$. This bypasses the singularity of ρ'' at zero. The finite difference (4.11) is applicable to the case of the median-interpolation problem; however, we remind the reader that at the time this article is written it is not known whether Method II applied to (4.10) would enjoy any convergence property.

Stopping criterion. We terminate a Newton iteration when the number of steps in a Armijo line search exceeds an upper bound `MAX_LINE_SEARCH` (chosen to be 10 in `MEstimatorInterp`.)

4.5. Numerical experiments.

Experiment I. $n = 4$,$(b_0, b_1, b_2, b_3, b_4) = (-1.2, 1.3, -0.9, 1.0, -0.8)$.

Experiment II. $n = 6$, $(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (0.05, 0.7, 0.6, -0.25, -0.38, -0.3, -1.5)$.

In both experiments, we take $I_i = [i, i+1]$. Each of (i)–(iv) in section 4.4 is applied to both datasets above; in the following discussion we label the corresponding subexperiments as I(i)–I(iv) and II(i)–II(iv), respectively. The left panels of Figure 1 depict the error curves $\log(||F(a^k)||)$ versus k; on the right panels we graph the midpoint, median, and smoothed median (with smoothing factor $\varepsilon = 0.1$) interpolants of the data.

While Method II applied to median interpolation converges in Experiment II(ii), the method clearly fails in Experiment I(ii). Nevertheless, the latter failure of convergence is remedied by Experiment I(iv). As mentioned, there is currently no theoretical guarantee that the Newton method with line search applied to the nonsmooth systems in (ii) would enjoy any global (or even local) convergence property. Moreover, one can see from Figure 1(a) and (c) that Newton Method II applied to the nonsmooth median interpolation does not seem to offer fast convergence.

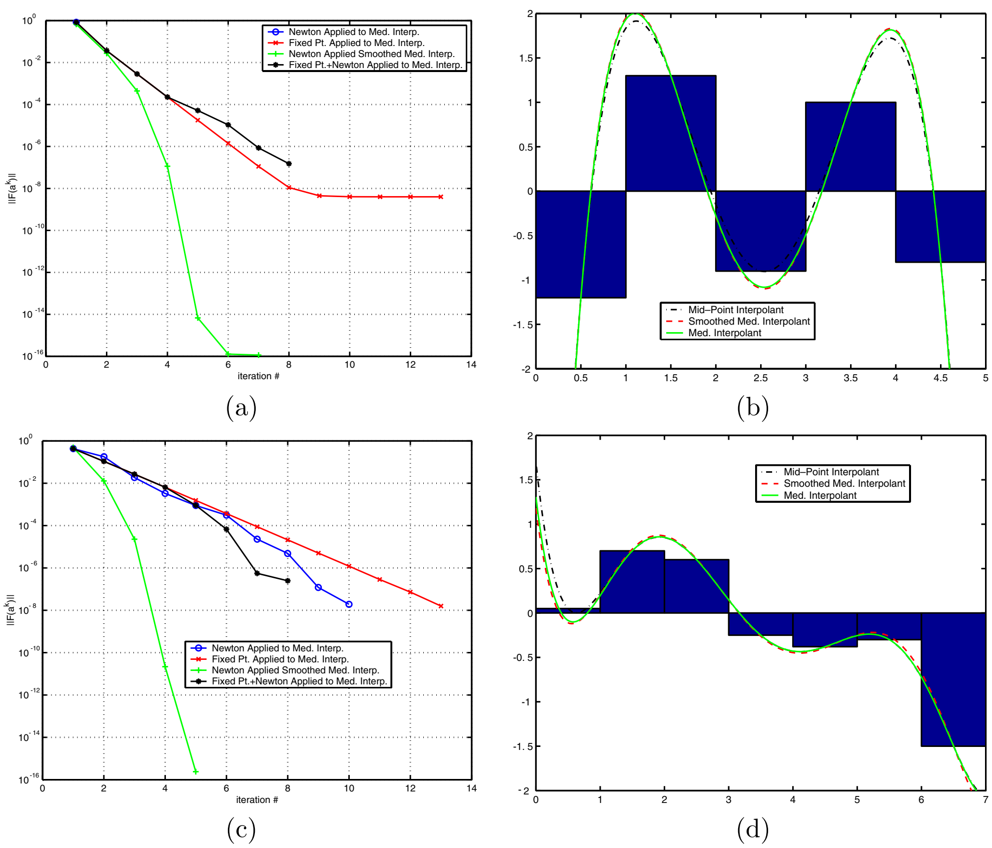

FIG. 1. *Error curves (left) and polynomial interpolants (right).*

On the other hand, Newton Method II applied to the smoothed median-interpolation problem exhibits superlinear convergence in Experiments I(i) and II(i), as expected from Theorem 4.3. As seen from Figure 1(a) and (c), the rates of convergence are noticeably faster than those in Experiment I(iii) and II(iii).

In summary the fixed-point algorithm used in `MedianInterp` for solving the median interpolation problem converges slower than Newton Method II applied to the smoothed median-interpolation problem. The latter method has a provable superlinear convergence property, while the former method has a conjectured linear convergence property. In terms of actual computational speed, one step of Newton iteration in our current implementation of `MEstimatorInterp` is much slower than one step of fixed-point iteration in `MedianInterp`; this is to be expected. Indeed, one can definitely improve the implementation of `MEstimatorInterp`.

During the revision of this article, we received the preprint by Qi [9], who studies the smoothness properties of $\mathbf{M}_\rho$ and F_ρ defined in section 4.1 and 4.2 in the case of the Huber M-Estimator (i.e., when ρ is given by (2.3).) The same article also illustrates a fundamental difficulty in the numerical solution of the median-interpolation problem: specifically, it is shown in section 4 of [9] that $F_{|\cdot|}$, unlike $M_{|\cdot|}$, is in general not even locally Lipschitz. This supports the slow convergence observed empirically in Experiments I(ii),(iv) and II(ii),(iv).

5. Nonlinear pyramid transforms based on M-estimators. Following the same strategy proposed in [3], one can now use the results in this paper to construct new nonlinear pyramid transforms: for a given signal $f : \mathbb{R} \to \mathbb{R}$, one measures the M-estimators of f over dyadic intervals $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$, performs coarse-to-fine prediction using local polynomial interpolation, which has been shown to be a well-posed procedure by Theorem 3.1, and then defines the pyramid coefficients based on the errors of such predictions.

Nonlinear subdivision operator. We first define a new class of nonlinear subdivision operators which will serve the purpose of coarse-to-fine prediction. Denote by $l(\mathbb{Z})$ the vector space of all real sequences defined on $\mathbb{Z}$. Let ρ be a convex function with the standard assumptions. Let $L \geq 1$ be an integer. For a given $y \in l(\mathbb{Z})$ and $h > 0$, define $S_{\rho,L;h}(y) \in l(\mathbb{Z})$ as follows.

1. Interpolation: for each $k \in \mathbb{Z}$, let $p_k \in \Pi_{2L}$ be the unique polynomial such that

$$m(p_k; [h(k+l), h(k+l+1)], \rho) = y_{k+l}, \quad l = -L, \dots, L.$$

2. Imputation:

$$\begin{aligned}(S_{\rho,L;h}(y))_{2k} &:= m(p_k; [hk, h(k+1/2)], \rho),\\ (S_{\rho,L;h}(y))_{2k+1} &:= m(p_k; [h(k+1/2), h(k+1)], \rho), \quad k \in \mathbb{Z}.\end{aligned}$$

Since $m(f; K, \rho) = m(f(T\cdot); T^{-1}(K), \rho)$ for any invertible affine map T, the operator $S_{\rho,L;h}$ is independent of the scale parameter h; thus we drop the subscript h and write $S_{\rho,L} : l(\mathbb{Z}) \to l(\mathbb{Z})$. It is worth mentioning that Theorem 3.1 implies that $S_{\rho,L}$ is a bounded operator on $l^\infty(\mathbb{Z})$.

Pyramid transform. For a continuous signal $f : \mathbb{R} \to \mathbb{R}$, a ρ, and an integer $L \geq 1$, we define an M-estimator interpolating pyramid transform of f, denoted by MeIPT$(f; L, \rho)$, as follows.

1. Formation of M-estimators of f over dyadic blocks:

$$m_{j,k} := m(f; I_{j,k}, \rho), \quad j = j_0, j_0+1, \dots, \ k \in \mathbb{Z}.$$

2. Coarse-to-fine prediction: $\widetilde{m}_{j+1} = S_{\rho,L}((m_{j,k})_k)$.
3. Formation of detailed coefficients: $d_{j,k} = m_{j,k} - \widetilde{m}_{j,k}$.

$$\text{MeIPT}(f; L, \rho) := \{(m_{0,k})_{k\in\mathbb{Z}}, (d_{1,k})_{k\in\mathbb{Z}}, (d_{2,k})_{k\in\mathbb{Z}}, \dots\}.$$

There is also an inversion process for recovering f from MeIPT$(f; L, \rho)$ based on sequentially reversing the steps above. For this purpose we need also the observation that

$$\lim_{j\to\infty} \left\| f - \sum_{k\in\mathbb{Z}} m(f; I_{j,k}, \rho) 1_{[2^{-j}k, 2^{-j}(k+1))} \right\|_{L^\infty} = 0$$

when f is a bounded continuous function.

For $j = j_0, j_0+1, j_0+2, \ldots,$

1. $f_j := \sum_{k\in\mathbb{Z}} m_{j,k} 1_{[2^{-j}k, 2^{-j}(k+1))}$.
2. Coarse-to-fine prediction: $\widetilde{m}_{j+1} = S_{\rho,L}((m_{j,k})_k)$.
3. Recovery of scale $j+1$ M-estimators of f: $m_{j+1,k} = \widetilde{m}_{j+1,k} + d_{j+1,k}, \quad k \in \mathbb{Z}$.

$$(f_j)_j \to f \text{ uniformly on compact sets.} \tag{5.1}$$

Discussion. In practice, however, one typically recovers an *estimate* of f from a certain *perturbed* version of MEIPT$(f; L, \rho)$; in this case the convergence in (5.1) has to be reexamined, and also the *stability* of MEIPT becomes a very important issue. While these open problems are beyond the scope of the current paper, based on Theorem 3.1 we expect MEIPT to be a decent tool for signal compression. Following the formulation and arguments in [3, Proof of P3, p. 1055], the transform coefficients in MEIPT(f) can be shown to have good *sparsity* when f is piecewise smooth; and the sparsity improves gracefully as the smoothness of f improves. Such a property is attributable to the accurate coarse-to-fine prediction power of the operator $S_{\rho,L}$, thanks to a polynomial exactness property of $S_{\rho,L}$ guaranteed by Theorem 3.1.

Robust pyramid transform based on smoothed medians. Combining the ideas in this and the last section, one can now construct MEIPT based on smoothed continuous medians. Such pyramid transforms had been found experimentally to be as robust against outliers as the median-interpolating pyramid transforms considered in [3]. This is to be expected by virtue of Proposition 4.4.

Acknowledgments. The authors are grateful to Dan Naiman for his discussion in the initial phase of the work. The authors would like to thank two referees for constructive comments on the original version of the paper, which have led to an improvement of the presentation.

REFERENCES

[1] D.P. BERTSEKAS, *Nonlinear Programming*, 2nd ed., Athena Scientific, Belmont, MA, 1999.
[2] J.E. DENNIS AND R.E. SCHNABEL, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice–Hall, Englewood Cliffs, NJ, 1983.
[3] D.L. DONOHO AND T. P.-Y. YU, *Nonlinear pyramid transforms based on median-interpolation*, SIAM J. Math. Anal., 31 (2000), pp. 1030–1061.
[4] D.L. DONOHO AND T. P.-Y. YU, *Nonlinear "Wavelet Transforms" Based on Median-Interpolation*, Technical report, Department of Statistics, Stanford University, Stanford, CA, 1997. Available online from http://stat.stanford.edu/reports/donoho/median.ps.Z.
[5] F. FACCHINEI AND J.S. PANG, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.
[6] T.N.T. GOODMAN AND T. P.-Y. YU, *Interpolation of medians*, Adv. Comput. Math., 11 (1999), pp. 1–10.
[7] P.J. HUBER, *Robust Statistics*, John Wiley and Sons, New York, 1981.
[8] J.M. ORTEGA AND W.C. RHEINBOLDT, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.
[9] L. QI, *Analytic Properties of the Huber and the Median M-Estimator Interpolation Problems*, preprint, Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, 2003.
[10] R.T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
[11] J. STOER AND R. BULIRSCH, *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980.