Smoothness Equivalence Properties of General Manifold-Valued Data Subdivision Schemes

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Abstract:
Based on a vector-bundle formulation, we introduce a new family of nonlinear subdivision schemes for manifold-valued data. Any such nonlinear subdivision scheme is based on an underlying linear subdivision scheme. We show that if the underlying linear subdivision scheme reproduces $\Pi_k$, then the nonlinear scheme satisfies an order $k$ proximity condition with the linear scheme. We also develop a new “proximity $\Rightarrow$ smoothness” theorem, improving the one in [12]. Combining the two results, we can conclude that if the underlying linear scheme is $C^k$ and stable, the nonlinear scheme is also $C^k$.

The family of manifold-valued data subdivision scheme introduced in this paper includes a variant of the log-exp scheme, proposed in [10], as a special case, but not the original log-exp scheme when the underlying linear scheme is non-interpolatory. The original log-exp scheme uses the same tangent plane for both the odd and the even rules, while the variant uses two different, judiciously chosen, tangent planes. We also present computational experiments that indicate that the original smoothness equivalence conjecture posted in [10] is unlikely to be true.

Our result also generalizes the recent results in [17, 16, 5, 6]. It uses only the intrinsic smoothness structure of the manifold and (hence) does not rely on any embedding or Lie group or symmetric space or Riemannian structure. In particular, concepts such as geodesics, log and exp maps, or projection from ambient space play no explicit role in the theorem. Also, the underlying linear scheme needs not be interpolatory.

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1 Introduction

In [10], the following log-exp scheme was proposed for the subdivision of data taking values on a Riemannian manifold or a Lie group:

$$y_{j+1,2i} = (Sy_{j})_{2i} = \exp_{y_{j,i}} \left( \sum_{\ell} a_{\ell} \log_{y_{j,i}} (y_{j,i+\ell}) \right), \quad y_{j+1,2i+1} = (Sy_{j})_{2i+1} = \exp_{y_{j,i}} \left( \sum_{\ell} b_{\ell} \log_{y_{j,i}} (y_{j,i+\ell}) \right).$$

Here $(a_{\ell})$ and $(b_{\ell})$ comes from the mask of a linear subdivision scheme $T$. It was conjectured by Donoho that this scheme satisfies a so-called smoothness equivalence property: if $T$ produces $C^k$ smooth curves, then so does $S$. While this conjecture had stimulated a number of studies and partial solutions, e.g. [17, 16, 5, 6] and the references therein, the conjecture remains unsolved.

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Geodesic-reproducing property. This log-exp scheme enjoys a geodesic-reproducing property. If the initial data consists of uniform samples of an arclength parameterized geodesics, then the log-exp scheme reproduces the geodesics. Related to this, it is clear that if the underlying Riemannian metric is flat, then the smoothness equivalence property is obviously satisfied. For example, applying the log-exp scheme $S$ on a cylinder is essentially the same as using the corresponding linear $T$ on the plane. This suggests that perhaps Donoho’s conjecture is true because (a) log and exp try to ‘locally flatten’ the manifold, and (b) subdivision is a local process. If one buys this intuition, one would speculate that the conjectured smoothness equivalence property is somewhat linked to the geodesic-reproducing property.

A 2005 Experiment. In an attempt to justify this intuition, we considered the case when $M = S^n$ and explored what happened when $\log_x(y)$ above was replaced by $g_x(y) := $ orthogonal projection of $y$ to $T_x M$.

Just like $\log_x$, $g_x$ is a local ($C^\infty$) diffeomorphism and has an inverse which we call $f_x$. And we considered

$$y_{j+1,2i} = (Sy)_j = f_y((\sum_\ell a_\ell g_{y,j}(y_{j,i}+\ell))),$$

$$y_{j+1,2i+1} = (Sy)_j = f_y((\sum_\ell b_\ell g_{y,j}(y_{j,i}+\ell))).$$

Unlike the log-exp scheme, this $g$-f scheme is not geodesic-reproducing. In the 2005 SIAM Conference on Geometric Design and Computing held at Pheonix, the first named author presented computational evidences that suggested the followings: When the underlying linear scheme is the $C^3$ quartic B-Spline scheme (with $(a_{-1}, a_0, a_1) = (5/16, 5/8, 1/16)$, $(b_{-1}, b_0, b_1) = (1/16, 5/8, 1/16)$), then

- the corresponding log-exp scheme is $C^3$, but
- the corresponding $g$-f scheme is only $C^2$ but not $C^3$.

For more details of this experiment, see [15, Chapter 4]. It was very evident from this experiment that the $g$-f scheme suffered from a breakdown of smoothness equivalence, while Donoho’s log-exp scheme, in this case, seemed to be doing fine in terms of smoothness equivalence. At the time of the conference, the authors were inclined to blame the breakdown to the fact that the $g$-f scheme was not geodesic-reproducing.

Surprisingly, all these turned out to be rather misleading.

1. First of all, Donoho’s original smoothness equivalence conjecture occurs to be not true. We shall present a set of computations that shows that the original log-exp scheme generally suffers a breakdown of smoothness equivalence at smoothness order 4 or 5. These computations require the use of variable precision arithmetic (vpa) in MAPLE. At the time of the 2005 SIAM conference, we were only using standard IEEE754 floating point computation in MATLAB and could not investigate smoothness equivalence breakdown at smoothness order higher than 3, as numerical differentiation of order 4 or higher is highly sensitive to floating point errors.
2. A simple twist of the arguments in [16] shows that Donoho’s conjecture is true when the underlying linear subdivision scheme is **interpolatory**. When the underlying linear subdivision scheme is non-interpolatory, we show in this article that a modified scheme of the following form can be proved to satisfy smoothness equivalence:

\[
y_{j+1,2i} = (S y_j)_{2i} = f_{y_{j,i}} \left( \sum_\ell a_\ell g_{y_{j,i}}(y_{j,i+\ell}) \right), \\
y_{j+1,2i+1} = (S y_j)_{2i+1} = f_{y_{j,i+1/2}} \left( \sum_\ell b_\ell g_{y_{j,i+1/2}}(y_{j,i+\ell}) \right),
\]

where \(y_{j,i+1/2}\) is a judiciously chosen point on the manifold. In fact this \(y_{j,i+1/2}\) can be conveniently computed using an **auxiliary interpolatory subdivision scheme**; see Section 3. The only property of \(f\) and \(g\) used in the proof is the smoothness of \((x, y) \mapsto f_x(y)\) and \((x, y) \mapsto g_y(y)\). Therefore, contrary to what the 2005 experiment suggested, smoothness equivalence property has little to do with geodesics or the log or exp maps, it only has to do with the smoothness of \(\log_x(y)\) and \(\exp_y(x)\) in \(x\) and \(y\).\(^1\) In particular, we can change \(\log_x\) and \(\exp_y\) to the \(g_x\) and \(f_y\) above, then, *as long as we change the base point for the odd rule from \(y_{j,i}\) to \(y_{j,i+1/2}\), we get full smoothness equivalence.*

### 1.1 Empirical breakdown of smoothness equivalence for Donoho’s log-exp scheme

An empirical way to check if a subdivision scheme \(S\) appears to be \(C^k\) is to plot the \(k\)-th order divided differences of the subdivision data \(v_j := S^j v_0\), i.e. \(2^k \Delta^k v_j\), and inspect visually if the plot looks continuous. Notice that, when the scheme is non-interpolatory, we only have \(v_{j,i} \approx (S^{\infty} v_0)(2^{-j} i)\), but under suitable stability conditions the following strong convergence can be shown:

\[
(2^k \Delta^k v_j)_i \to (S^{\infty} v_0)^{(k)}(2^{-j} i), \quad j \to \infty.
\]  

(When \(S\) is a linear scheme, this is well-known, see [11, 4]. When \(S\) is nonlinear, it can be shown via a proximity condition with a stable linear scheme. The latter forms part of the proof of Theorem 2.4.)

When both \(k\) and \(j\) are sufficiently large, then the computed values of \((2^k \Delta^k v_j)_i\) are very inaccurate due to floating point errors. Fortunately, we can use \texttt{vpa} provided by MAPLE to improve accuracy. In Figure 2, we plot \((2^9 \Delta^6 S^9 \delta)_i\) on \(2^{-9}\mathbb{Z}\), computed using \texttt{vpa} with 16, 18, \ldots, 21 digits. Here, \(S\) is the linear degree 8 B-spline subdivision scheme. This scheme is \(C^7\), so the 6-th order divided differences is continuously differentiable. The plots clearly illustrate how \texttt{vpa} is crucial for illustrating the smoothness of the 6-th order divided differences.

![Figure 2: Plots of the 6th derivative of the degree 8 B-Spline based on taking the 6th order divided differences of the level 9 subdivision data, computed using variable precision arithmetic (vpa) in Maple with 16, 18, \ldots, 21 decimal digits. Note: vpa with 16 digits is comparable to standard IEEE 754 floating point computation, and is insufficient in this particular case to give an accurate plot of the 6th order divided differences. With vpa, one can compute with higher accuracy at a reasonable computational speed.](image)

Using this naive divided difference approach, we now compare Donoho’s log-exp scheme with a modified log-exp scheme, the latter is a special case of the general scheme to be proposed in Section 3. For now, all we

\(^1\)Since \(\exp_y(x)\) is defined by geodesics, and geodesics are described by ODEs, the smoothness of \((x, y) \mapsto \exp_y(x)\) comes from the smoothness of the solutions of ODEs and the smooth dependence of the solutions on initial data. See [8, Chapter XIV] for a modern functional analytic proof of this nontrivial classical result.
need to know about this modified log-exp scheme is that it has a **provable** smoothness equivalence property. In this computational experiment, the manifold is the 2-sphere with the Riemannian metric inherited from $\mathbb{R}^3$; the linear scheme is the degree 7 B-spline subdivision scheme.

It can be shown, based on the arguments in [12], that the original log-exp scheme is $C^2$ in this case. The same is true for the modified log-exp scheme. These are illustrated by the first row of Figure 3. The modified log-exp scheme is proved to be $C^6$, a fact that is also evident from the plots. Also very evident is that the original log-exp scheme fails to be $C^5$; in fact the plots also suggest that the scheme may not even be $C^4$ (see the 3rd row.)

As the control experiment in Figure 2 shows, when a divided difference plot appears to be not smooth, it can be due to roundoff errors and/or the non-smoothness of the underlying function. However, by generating the same plot with increasing precision using vpa, the former effect goes away and we can reliably judge whether the underlying function is smooth or not. For instance, look at the leftmost panel of the 4th row of Figure 3. The 5th order divided difference plots (of level $j = 8$ subdivision data) for both schemes, computed using only 8 digits, exhibit no smoothness whatsoever, but as the precision increases, it becomes evident that the modified scheme is $C^5$ smooth but the original scheme is not. The roughness observed in the original scheme is **not** due to roundoffs.

A Maple spreadsheet that reproduces this experiment can be found in 

http://www.math.drexel.edu/~tyu/DonohoConjectureBreakDown

## 2 Proximity Conditions and Smoothness

In this section, we offer a new “Proximity $\Rightarrow$ Smoothness” theorem. This kind of theorems was first proposed in [13, 12]. Compared to a corresponding theorem in [12], the new version here gets rid of certain unnatural assumptions pertaining to the underlying linear scheme. Other related papers that use the same notion of proximity conditions are [14, 7].

Throughout this section, we let $Z^+ := \{0\} \cup \mathbb{N}$. For any sequence $x = (x_i)_i$ in an Euclidian space, let $|x|_\infty := \sup_i \|x_i\|_2$.

For any $M \subseteq \mathbb{R}^N$ and $\delta > 0$, let

$$X_M := \{x : Z \rightarrow M \mid |\Delta x|_\infty < \infty\}$$

and

$$X_{M,\delta} := \{x : Z \rightarrow M \mid |\Delta x|_\infty < \delta\}.$$ 

For $j \in \mathbb{N}$, let

$$\Gamma_j := \left\{ \gamma \in Z^+, \sum_{i=1}^j \gamma_i \geq 2, \sum_{i=1}^j i \gamma_i = j + 1 \right\}.$$ 

Note that $2 \leq |\gamma| := \gamma_1 + \cdots + \gamma_j \leq j + 1$ for any $\gamma \in \Gamma_j$. Obviously, $\Gamma_j$ is a finite set for every $j \in \mathbb{N}$. For any $x \in X_M$, let

$$\Omega_j(x) := \sum_{\gamma \in \Gamma_j} \prod_{i=1}^j |\Delta^i x|_\infty^{\gamma_i}.$$ 

For example, $\Omega_1(x) = |\Delta x|_\infty^2$ and $\Omega_2(x) = |\Delta x|_\infty^3 + |\Delta x|_\infty |\Delta^2 x|_\infty$.

Since $|\Delta^k x|_\infty \leq 2|\Delta^{k-1} x|_\infty$ for any $k \in \mathbb{N}$, it follows that for any $j \in \mathbb{N}$, there exists a constant $\alpha_j$ such that for any $x \in X_{M,\delta}$,

$$\Omega_{j+1}(x) \leq \alpha_j \Omega_j(x). \quad (2.1)$$

**Lemma 2.1.** Let $S : X_{M,\delta} \rightarrow X_M$. Let $\tilde{S}$ be a convergent linear subdivision operator with dilation factor $D$. Suppose there exists $C_1 > 0$ such that

$$|Sx - \tilde{S}x|_\infty \leq C_1 \Omega_1(x), \quad \forall x \in X_{M,\delta},$$

then there exist $\delta' > 0$, $C > 0$ and $\beta > 0$ such that $S^jx$ is well-defined and $|\Delta S^j x|_\infty \leq CD^{-j\beta} |\Delta x|_\infty$ for any $j \in \mathbb{N}$ and $x \in X_{M,\delta'}$. 

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Proof: See Section A.1.

For any sequence $x : \mathbb{Z} \to \mathbb{R}^N$, $n \in \mathbb{Z}^+$, and $D > 1, D \in \mathbb{N}$, we define $\mathcal{F}_D^n(x)$ to be the piecewise linear function with $\mathcal{F}_D^n(x)(iD^{-n}) = x_i$. For fixed $D$ and $n$, $\mathcal{F}_D^n$ can be viewed as a linear map from the space of sequences $x$ to the space of piecewise linear functions; moreover, we have

$$|\mathcal{F}_D^n(x) - \mathcal{F}_D^n(\bar{x})|_\infty = |x - \bar{x}|_\infty, \quad \forall \, x, \bar{x}. \quad (2.2)$$

For a subdivision operator $S : \mathcal{X}_{M,\delta} \to \mathcal{X}_M$ with dilation factor $D$, if there exists $\delta' > 0$ such that $\mathcal{F}_D^n(S^n x)$ converges to a $C^k (k \in \mathbb{Z}^+)$ function as $n \to \infty$ for any $x \in \mathcal{X}_{M,\delta'}$, then we say $S$ is $C^k$. To prove $S$ is $C^k (k \in \mathbb{Z}^+)$, it suffices to show that there exists a $\delta' > 0$ such that for any $x \in \mathcal{X}_{M,\delta'}$ and $j = 0, 1, \cdots, k$, $\mathcal{F}_D^n(D^{jn}\Delta_j S^n x)$ converges uniformly as $n \to \infty$. See [3, Theorem 4.1].

We need the following lemma. Its proof can be found in [2] or [12, Lemma 1].

Lemma 2.2. Let $\tilde{S}$ be a convergent linear subdivision operator with dilation factor $D$. Then there exists a constant $C > 0$ such that for any sequence $x$ and $n \in \mathbb{N}$,

$$|\mathcal{F}_D^n(\tilde{S} x) - \mathcal{F}_D^n(x)|_\infty \leq C|\Delta x|_\infty.$$

We first prove the following basic “proximity $\Rightarrow$ continuity” result.

Theorem 2.3. Let $S : \mathcal{X}_{M,\delta} \to \mathcal{X}_M$ be a subdivision operator and $\tilde{S}$ be a convergent linear subdivision operator. Suppose $S$ and $\tilde{S}$ have the same dilation factor and there exists $C_1 > 0$ such that

$$|Sx - \tilde{S} x|_\infty \leq C_1 \Omega_1(x), \quad \forall \, x \in \mathcal{X}_{M,\delta}.$$  

(2.3)

Then $S$ is $C^0$.

Proof: Suppose $S$ and $\tilde{S}$ have dilation factor $D$. It follows from Lemma 2.1 that there exists $C > 0$, $\beta > 0$ and $\delta' > 0$ such that $S^n x$ is well-defined and $|\Delta S^n x|_\infty \leq CD^{-n\beta}|\Delta x|_\infty$ for any $n \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$. Combining with Lemma 2.2, (2.2) and (2.3), we have for any $n \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$,

$$|\mathcal{F}_D^n(S^n x) - \mathcal{F}_D^n(S^{n+1} x)|_\infty \leq |\mathcal{F}_D^n(S^n x) - \mathcal{F}_D^{n+1}(\tilde{S} S^n x)|_\infty + |\mathcal{F}_D^{n+1}(\tilde{S} S^n x) - \mathcal{F}_D^n(S^{n+1} x)|_\infty \leq C|\Delta S^n x|_\infty + |S^{n+1} x - \tilde{S} S^n x|_\infty \leq C|\Delta S^n x|_\infty + C_1|\Delta S^n x|_\infty^2 \leq C CD^{-n\beta}|\Delta x|_\infty + C_1 C^2 D^{-2n\beta}|\Delta x|_\infty^2 \leq C C D^{-n\beta} \delta' + C_1 C^2 D^{-2n\beta} \delta'^2.$$

Therefore $\mathcal{F}_D^n(S^n x)$ is a Cauchy sequence. Note $\mathcal{F}_D^n(S^n x)$ is a $C^0$ function for any $n \in \mathbb{N}$. Hence $\mathcal{F}_D^n(S^n x)$ converges to a $C^0$ function as $n \to \infty$. This means $S$ is $C^0$.

The final goal of this section is to extend Theorem 2.3 to higher order smoothness. To prove this theorem we need to first develop a few auxiliary lemmas, see Appendix A.2.

Theorem 2.4. Let $S$ be a subdivision operator defined on $\mathcal{X}_{M,\delta}$ and $\tilde{S}$ be a $C^k$ ($k \geq 1$) linear $L_\infty$-stable subdivision operator. Suppose $S$ and $\tilde{S}$ have the same dilation factor. If there exists $C_1 > 0$ such that for any $x \in \mathcal{X}_{M,\delta}$ and $j = 1, \cdots, k$

$$|\Delta^{j-1} S x - \Delta^{j-1} \tilde{S} x|_\infty \leq C_1 \Omega_j(x).$$

Then $S$ is $C^k$.

Proof: Suppose the dilation factor of $S$ and $\tilde{S}$ is $D$. Since $\tilde{S}$ is $C^k$ ($k \geq 1$), linear and $L_\infty$-stable, it follows (e.g. [11, 4]) that there exist $C_{1, \cdots, C_{k+1}} \geq 1$ and $\mu \in [0, 1)$ such that

$$|\Delta^j \tilde{S} x|_\infty \leq C_{1} D^{-j\mu}|\Delta^j x|_\infty, \quad j = 1, \cdots, k; \quad (2.4)$$

$$|\Delta^{k+1} \tilde{S} x|_\infty \leq C_{k+1} D^{(-j-1+\mu)n}|\Delta^{k+1} x|_\infty.$$ 

(2.5)
We can artificially modify the above constants such that
\[ C^+_{j} < C_{j+1}, \quad j = 1, \ldots, k. \] (2.6)
Since \( \mu < 1 \), there exists \( m \in \mathbb{N} \) such that
\[ \bar{\mu}_{k+1} := \mu + \frac{\log_{\tilde{D}} C_{k+1}}{m} < 1. \]
Define \( \bar{\mu}_j = \frac{\log_{\tilde{D}} C_j}{m} \) for \( j = 1, \ldots, k \). Then it follows from (2.6) that
\[ \bar{\mu}_j < \frac{\bar{\mu}_{j+1}}{j+1}, \quad j = 1, \ldots, k. \]

Let \( T := S^m \) and \( \tilde{T} := \tilde{S}^m \) be two new subdivision operators. Then both of them have dilation factor \( \tilde{D} := D^m \) and \( \tilde{T} \) is \( C^k \), linear and \( L_\infty \)-stable. Hence \( \tilde{T} \) has derived subdivision operators \( \tilde{T}_1, \ldots, \tilde{T}_k \). It follows from Lemma A.3 that there exists \( C_m > 0 \) such that
\[ |\Delta^{-1}T x - \Delta^{-1}\tilde{T}_j x|_\infty \leq C_m \Omega_j(x), \quad \forall x \in X_{\mathcal{M}, \delta}, j = 1, \ldots, k. \]

It follows from (2.4)(2.5) that
\[ |\Delta^{j}T x|_\infty \leq \Delta^{-j+\bar{\mu}_1}|\Delta^j x|_\infty, \quad j = 1, \ldots, k + 1. \]
Choose
\[ 0 < \epsilon < \min \left( 1 - \bar{\mu}_1, \min_{1 \leq j \leq k} \left( \frac{\bar{\mu}_{j+1}}{j+1} - \bar{\mu}_j \right) \right). \]
Define \( \mu_1 = \bar{\mu}_1 + \epsilon, \mu_2 = \bar{\mu}_2, \ldots, \mu_{k+1} = \bar{\mu}_{k+1} \). Then \( \mu_j \in (0, 1) \) for \( j = 1, \ldots, k + 1 \) and
\[ \mu_j < \frac{\mu_{j+1}}{j+1}, \quad j = 1, \ldots, k. \]
It follows from Lemma A.1 that there exist \( 0 < \delta' \leq \delta \) and polynomials \( P_1, \ldots, P_{k+1} \) such that for any \( n \in \mathbb{N} \) and \( x \in X_{\mathcal{M}, \delta'} \),
\[ |\Delta^{j}T^m x|_\infty \leq \tilde{D}^{(-j+\mu_1)n} P_j(n)|\Delta^j x|_\infty, \quad j = 1, \ldots, k + 1. \]
It follows from Lemma A.2 that \( T = S^m \) is \( C^k \). Therefore \( \tilde{T} \) is a Deslauriers-Dubuc subdivision operator.

\[ \textbf{Remark:} \] Theorem 2.3 is similar to [13, Theorem 3] while Theorem 2.4 is similar to [12, Theorem 6]. But in each case our theorem is more general. For example, [12, Theorem 6] applies when \( \bar{S} \) is a B-spline subdivision operator but not when \( \tilde{S} \) is a Deslauriers-Dubuc subdivision operator.

3 A New Nonlinear Subdivision Scheme for Manifold-Valued Data

In this section we introduce the general subdivision scheme for manifold-valued data as promised in Section 1. The definition assumes some familiarity with basic manifold theory. Those who do not want to deal with manifold concepts can basically skip this section and directly study the localized version in (4.1)-(4.3).

Let \( M \) be a smooth manifold of dimension \( n \). We let \( T \) be a linear, not necessarily interpolatory, subdivision operator defined by
\[ (Ty)_{2i} = \sum_{\ell} a_{\ell} y_{i+\ell}, \quad (Ty)_{2i+1} = \sum_{\ell} b_{\ell} y_{i+\ell}. \] (3.1)
Our general nonlinear subdivision scheme \( S \) for \( M \)-valued data will be derived from this linear scheme \( T \) as well as an auxiliary linear interpolatory subdivision scheme \( \tilde{T} \) defined by
\[ (\tilde{T}y)_{2i} = y_i, \quad (\tilde{T}y)_{2i+1} = \sum_{\ell} c_{\ell} y_{i+\ell}. \] (3.2)
To prepare for our definition of $S$, we need also two pairs of smooth maps $(f, g)$ and $(\hat{f}, \hat{g})$ defined as follows. Let $\pi : V \to M$ and $\hat{\pi} : \hat{V} \to M$ be two vector bundles over $M$, with ranks $N$ and $N$ respectively. Write

$$V(x) := \pi^{-1}(x) \quad \text{and} \quad \hat{V}(x) := \hat{\pi}^{-1}(x)$$

as the fibres of $V$ and $\hat{V}$ at $x$, respectively. Recall that $V$ has by definition a differentiable structure of dimension $n + N$. Elements in $V$ will be denoted by a tuple of the form $(x, v)$, where $x \in M$ and $v \in V(x)$.

Similar comments apply to $\hat{V}$. Assume that, for every $x \in M$, there are open neighborhoods $U_x$ and $\hat{U}_x$ of $x$, and associated smooth mappings

$$g_x : U_x \to V(x), \quad \hat{g}_x : \hat{U}_x \to \hat{V}(x).$$

It is crucial for us to also assume that the mappings $(x, y) \mapsto g_x(y)$ and $(x, y) \mapsto \hat{g}_x(y)$ are jointly smooth in $x$ and $y$. For this purpose, we need to first assume that

$$U := \{(x, y) : x \in M, y \in U_x\} \quad \text{and} \quad \hat{U} := \{(x, y) : x \in M, y \in \hat{U}_x\}$$

are open in $M \times M$. This openness assumption allows us to sensibly talk about the smoothness of the maps $g$ and $\hat{g}$ below. It also implies that $U_x$ cannot get arbitrarily small locally in a sense to be made precise in the proof of Lemma 3.2.

Let

$$g : U \to V, \quad \hat{g} : \hat{U} \to \hat{V},$$

be defined by

$$g(x, y) = (x, g_x(y)), \quad \text{and} \quad \hat{g}(x, y) = (x, \hat{g}_x(y));$$

we assume that both maps are smooth. Without loss of generality, we can also assume that

$$g_x(x) = 0 \quad \text{and} \quad \hat{g}_x(x) = 0.$$  

(3.5)

Let $E_x$ (resp. $\hat{E}_x$) be an open set in $V(x)$ (resp. $\hat{V}(x)$) that contains $g_x(U_x)$ (resp. $\hat{g}_x(\hat{U}_x)$) such that

$$E := \{(x, v) : x \in M, v \in E_x\}$$

is open in $V$ (resp. $\hat{E} := \{(x, v) : x \in M, v \in \hat{E}_x\}$ is open in $\hat{V}$.) Then, let

$$f_x : E_x \to M, \quad \hat{f}_x : \hat{E}_x \to M,$$

be two smooth maps that satisfy

$$f_x(g_x(y)) = y, \quad \forall x \in M, y \in U_x, \quad \hat{f}_x(\hat{g}_x(y)) = y, \quad \forall x \in M, y \in \hat{U}_x.$$  

(3.8)

We then assume also that

$$f : E \to M \quad \text{and} \quad \hat{f} : \hat{E} \to M$$

are smooth; this means $f_x(y)$ and $\hat{f}_x(y)$ are jointly smooth in $x$ and $y$.

We are now ready to define $S$.

**Definition 3.1.** Given $T, \hat{T}, (f, g)$ and $(\hat{f}, \hat{g})$ defined above, a nonlinear subdivision scheme $S$ is defined by:

$$(Sx)_{2i} := f_{x_i}(\sum_\ell a_\ell g_{x_i}(x_{i+\ell})), \quad (Sx)_{2i+1} := f_{x_{i+1/2}}(\sum_\ell b_\ell g_{x_{i+1/2}}(x_{i+\ell}))$$

(3.9)

where

$$x_{i+1/2} = \hat{f}_{x_i}(\sum_\ell c_\ell \hat{g}_{x_i}(x_{i+\ell})).$$  

(3.10)

We can, and we will, view (3.10) as an interpolatory subdivision scheme for $M$-valued data:

$$(\hat{S}x)_{2i} = x_i, \quad (\hat{S}x)_{2i+1} = \hat{f}_{x_i}(\sum_\ell c_\ell \hat{g}_{x_i}(x_{i+\ell})).$$

(3.11)
Examples of this abstract scheme include:

- \( V = TM \) (the tangent bundle.) \( M \) is a Riemannian manifold or a Lie group, \( g_x = \log_x \) and \( f_x = \exp_x \).
  In this case \( S \) is a **variant** of the log-exp scheme proposed in [10]. Note that \((\hat{V}, \hat{f}, \hat{g})\) can be chosen arbitrarily as long as they satisfy the basic assumptions. See Section 1.1 for a comparison of the original log-exp scheme with this variant.

- \( V = M \times \mathbb{R}^n \) (a trivial bundle.) \( g_x(y) = (x, i(y)), \) where \( i : M \to \mathbb{R}^n \) is an embedding of \( M \) into \( \mathbb{R}^n, \) \( f_x(v) = i^{-1}(\text{the point in } i(M) \text{ closest to } v). \) In this case, this is the **closest point projection scheme** studied in [13, 17, 6]. Note: Mapping a point from the ambient space of a regular surface to the closest point on the surface is well-defined on a tubular neighborhood of the surface (see, e.g., [1].) This tubular neighborhood of \( i(M) \) furnishes the natural domain of \( f_x. \) In this case, both \( g_x \) and \( f_x \) are essentially independent of \( x; \) therefore the auxiliary interpolatory scheme \( \hat{T} \) does not actually play a role in the definition of \( S. \)

We shall prove that if the initial sequence \((x_i)\) is dense enough, then the subdivision processes \( S^j x, \hat{S}^j x,\) \( j = 0, 1, 2, \ldots \), are well-defined. In virtue of Lemma 2.1, this can be established by proving (a) \( S \) and \( \hat{S} \) are well-defined in one step (i.e. \( Sx \) and \( \hat{S}x \) are well-defined for dense enough sequences \( x \)), and (b) proximity conditions between \( S \) and \( T \) and between \( S \) and \( \hat{T}. \)

Proximity conditions will be the main theme of the next section. To prove the one-step well-definedness, we need the following.

**Lemma 3.2.** Let \( M \) be a topological manifold, and \( V \) be a vector bundle over \( M. \) Let \((W, \phi)\) be any chart of \( M \) such that the vector bundle \( V \) is trivial on \( W. \) Let \( K \) be any compact subset of \( W. \) Let \( f \) and \( g \) be defined as above, except that they are only assumed to be continuous (instead of smooth, as we do not assume in this lemma that there is a differentiable structure on \( M). \) Let \((w_i)\) be any finitely supported sequence. (Assume, without loss of generality, that \( 0 \in \text{support}(w).\) Then for every \( \varepsilon > 0, \) there exists \( \delta := \delta(W, \phi, K, W, f, g) > 0 \)

such that for any \((x_\ell)_{\ell \in \text{support}(w)}, x_\ell \in K,\) with

\[
\max_{\ell \in \text{support}(w)} \| \phi(x_\ell) - \phi(x_0) \| < \delta,
\]

\( f_{x_0}(\sum_\ell w_\ell g_{x_0}(x_\ell)) \) is well-defined and falls inside the chart \( W, \) moreover

\[
\| \phi(f_{x_0}(\sum_\ell w_\ell g_{x_0}(x_\ell))) - \phi(x_0) \| < \varepsilon.
\]

**Proof:** By local triviality, the chart \((W, \phi)\) induces an isomorphism

\[
i : \{(x, v) \in V : x \in W, v \in V(x)\} \to \phi(W) \times \mathbb{R}^N.
\]

We also write \( i_x := i(x, \cdot) : V(x) \to \mathbb{R}^N. \) Note that \( i(x, v) = (\phi(x), i_x(v)). \) Without loss of generality, we can assume that \( i_x(0) = 0. \)

1° The openness assumption on \( U = \{(x, y) : x \in M, y \in U_x\} \) implies that \( \{(\phi(x), \phi(y)) : x \in K, y \in U_x \cap W\} \) is open in \( \phi(K) \times \mathbb{R}^n. \) Since \( x \in U_x \) for all \( x, \) \( \{(\phi(x), \phi(y)) : x \in K, y \in U_x \cap W\} \) is an open set in \( \phi(K) \times \mathbb{R}^n \) which contains the ‘line’ \( \phi(K) \times \{0\}, \) and therefore also contains, in virtue of the compactness of \( K \) and the tube lemma [9], a ‘tube’ \( \phi(K) \times B_{\mathbb{R}^n}(0, r). \) Hence, for every \( x \in K, \) \( \phi(U_x \cap W) \) contains an open ball of radius \( r > 0. \) Of course, this \( r \) can be chosen such that the closure of \( B_{\mathbb{R}^n}(\phi(x), r) \) is also contained in \( \phi(U_x \cap W). \)

So we can write \( g \) in local coordinates as a map of the form

\[
g_{(W, \phi)} : \phi(K) \times B_{\mathbb{R}^n}(0, r) \to \phi(K) \times \mathbb{R}^N, \quad (x, y) \to i(g\left(\phi^{-1}(x), \phi^{-1}(x + y)\right)).
\]

2° We denote by \( B_{\mathbb{R}^n}(x, r), \) or simply \( B(x, r) \) when the dimension is clear from the context, an open ball in \( \mathbb{R}^n \) centered at \( x \in \mathbb{R}^n \) with radius \( r > 0. \)
Since \(g(W, \phi)(x, y) = (x, v)\), the only information in this map is the part that maps \((x, y)\) to \(v\). We denote this map by
\[
g : \phi(K) \times B_{\mathbb{R}^n}(0, r) \to \mathbb{R}^N. \tag{3.14}
\]
Note that \(g(x, 0) = 0\) by (3.5). We also write
\[
g_x := g(x, \cdot) : B_{\mathbb{R}^n}(0, r) \to \mathbb{R}^N.
\]

2° We claim that there exists a \(r' > 0\) such that
\[
B_{\mathbb{R}^n}(0, r') \subseteq \bigcup_{x \in K} \{ f_x^{-1}(\phi^{-1}(B(\phi(x), r))) \}, \forall x \in K. \tag{3.15}
\]
This is again proved using the tube lemma. Consider the continuous \(F : E \to M \times M\) defined by \(F(x, v) = (x, f_x(v))\). We have
\[
\{(x, y) : x \in K, y \in \phi^{-1}(B(\phi(x), r))\} \quad \text{open} \quad K \times W \subset M \times M,
\]
\[
F^{-1}\left(\{(x, y) : x \in K, y \in \phi^{-1}(B(\phi(x), r))\}\right) \quad \text{open} \quad \bigcup_{x \in K} E_x \subset \bigcup_{x \in K} V(x),
\]
therefore
\[
i \left( F^{-1}\left(\{(x, y) : x \in K, y \in \phi^{-1}(B(\phi(x), r))\}\right) \right) \quad \text{open} \quad \phi(K) \times \mathbb{R}^N.
\]
But the set \(i \left( F^{-1}\left(\{(x, y) : x \in K, y \in \phi^{-1}(B(\phi(x), r))\}\right) \right)\) contains the ‘line’ \(\phi(K) \times \{0\}\) and hence also a ‘tube’ \(\phi(K) \times B_{\mathbb{R}^n}(0, r')\). This is equivalent to our claim (3.15).

Using (3.15), we can write \(f\) in the local coordinates as:
\[
f : \phi(K) \times B_{\mathbb{R}^n}(0, r') \to B_{\mathbb{R}^n}(0, r), \quad (x, v) \mapsto \phi\left( f\left( i^{-1}(x, v) \right) \right) - x. \tag{3.16}
\]
Define also
\[
f_x := f(x, \cdot) : B_{\mathbb{R}^n}(0, r') \to B_{\mathbb{R}^n}(0, r). \tag{3.17}
\]

3° Let \(\varepsilon > 0\) be given. Since \(\phi(K) \times B_{\mathbb{R}^n}(0, r')\) is compact, \(f\) is uniformly continuous, so there exists a \(\delta' \in (0, r']\) such that \((x, y) \in \phi(K) \times B_{\mathbb{R}^n}(0, \delta') \Rightarrow \|f(x, v)\| < \varepsilon\). Similarly, there exists a \(\delta \in (0, r]\) such that \((x, y) \in \phi(K) \times B_{\mathbb{R}^n}(0, \delta) \Rightarrow \|g_x(y)\| < \delta'/\sum_{\ell} |w_{\ell}|\).

Therefore if \(x_{\ell} \in \phi(K)\) satisfies \(\|x_{\ell} - x_0\| < \delta\), then \(\|g_{x_0}(x_{\ell})\| < \delta'/\sum_{\ell} |w_{\ell}|\), which implies \(\|\sum_{\ell} w_{\ell}\delta_{x_0}(x_{\ell})\| < \delta', \) which then implies \(\|\sum_{\ell} w_{\ell}\delta_{x_0}(x_{\ell})\| < \varepsilon\).

\section{Establishing Proximity Conditions}

Since our subdivision schemes \(T\) and \(\hat{T}\) act locally to the data, there is a minimal set of indices, which we denote here by \(\mathcal{N}(0, i)\), such that the level zero data \(x_{0,k}, k \in \mathcal{N}(0, i)\), is enough to determine the limit functions \(T^\infty y\) and \(\hat{T}^\infty y\) at the whole interval \([i - 1, i + 1]\). In order to analyze the scheme in Definition 3.1 is smooth, it suffices to show that it produces a smooth limit on \([i - 1, i + 1]\) for every \(i \in \mathbb{Z}\). We shall assume that for every \(i \in \mathbb{Z}\) there is a chart \((W, \phi)\) of the manifold that trivializes both vector bundles \(V\) and \(\hat{V}\) and also contains the data \(x_{0,k}, k \in \mathcal{N}(0, i)\). This forms part of our ‘dense enough’ assumption on the initial manifold-valued data. With this assumption, we can rewrite the scheme locally in terms of local coordinates. By shift invariance, it suffices to analyze only the case of \(i = 0\).

Following the proof of Lemma 3.2, given a chart \((W, \phi)\) on which both \(V\) and \(\hat{V}\) are trivial, we can then write \(g, \tilde{g}, f, \tilde{f}\) in local coordinates as
\[
g : K \times B_{\mathbb{R}^n}(0, r) \to \mathbb{R}^N, \quad f : K \times B_{\mathbb{R}^n}(0, r') \to B_{\mathbb{R}^n}(0, r), \tag{4.1}
\]
\[
\tilde{g} : K \times B_{\mathbb{R}^n}(0, r) \to \hat{\mathbb{R}}^N, \quad \tilde{f} : K \times B_{\mathbb{R}^n}(0, r') \to B_{\mathbb{R}^n}(0, r). \tag{4.2}
\]
Here $K \subset \phi(W)$ is a compact set in $\mathbb{R}^n$; this $K$ is the $\phi(K)$ in Lemma 3.2. Notice that we abuse notations here by using the same symbols to denote the localized version of $g$, $\hat{g}$, $f$ and $\hat{f}$. By construction, $g(x, 0) = 0$, $\hat{g}(x, 0) = 0$, $f(x, 0) = 0$, $\hat{f}(x, 0) = 0$, and

$$f(x, g(x, y)) = y = \hat{f}(x, \hat{g}(x, y)) \quad \forall x \in K, y \in B_{R^c}(0, r).$$

By Lemma 3.2, if $x_\ell \in K$, $\ell \in \mathcal{N}(0, 0)$ satisfies $\|x_\ell - x_0\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$ then for all relevant indices $h$,

$$(Sx)_{2h} = x_h + f\left(x_h, \sum_\ell a_\ell g(x_h, x_{h+\ell} - x_h)\right),$$

$$x_{h+1/2} = (Sx)_{2h+1} = x_h + \hat{f}\left(x_h, \sum_\ell b_\ell \hat{g}(x_h, x_{h+\ell} - x_h)\right),$$

$$\text{(4.3)}$$

are well-defined and all fall within the ball $B(x_0, r)$. Notice that (4.3) is our original subdivision schemes in Definition 3.1 written in local coordinates.

The goal of this section is to prove the following two proximity conditions:

**Theorem 4.1.** Assume that the polynomial reproduction orders of $\hat{T}$ and $T$ are at least $k$. Then there exists a constant $C > 0$ such that for any sufficient dense sequence $(x_\ell)_{\ell \in \mathcal{N}(0, 0)}$,

$$\|\hat{S}x - \hat{T}x\| \leq C \Omega_{k'} ((x_\ell)_\ell)$$

and

$$\|\Delta^{k'-1}(Sx - Tx)\| \leq C \Omega_{k'} ((x_\ell)_\ell)$$

for all $k' = 1, \ldots, k$.

**Proof:** We prove the theorem for $k' = k$, the proof for any smaller $k'$ is identical. We divide the proof into a number of steps. The first four steps serve as a preparation for the proof of both parts of the theorem. In $1^o-4^o$ below, $(w_\ell)_\ell$ is any finitely supported sequence with $\sum w_\ell = 1$ and, in particular, can be any one of the $(a_\ell), (b_\ell)$ or $(c_\ell)$ from the masks of $T$ and $\hat{T}$, and $f$ and $g$ may refer to either the $f$ and $g$ in (4.1) or else the $\hat{f}$ and $\hat{g}$ in (4.2). Step $5^o$ finishes the proof of (4.4). Steps $6^o-9^o$ deal with (4.5).

The assumption that $\hat{T}$ reproduces $\Pi_k$ (= the space of polynomials with degree $\leq k$) is equivalent to

$$\sum_\ell c_\ell \pi(\ell) = \pi(1/2), \quad \forall \pi \in \Pi_k;$$

that $T$ reproduces $\Pi_k$ is equivalent to

$$\sum_\ell a_\ell \pi(\ell + 1/2) = \sum_\ell b_\ell \pi(\ell), \quad \forall \pi \in \Pi_k.$$  

(4.7)

$1^o$ Write

$$F^{(m)} := \frac{1}{m!} d^m f|_{(x_0, 0)}, \quad G^{(m)} := \frac{1}{m!} d^m g|_{(x_0, 0)}.$$  

(4.8)

These are multi-linear maps with

$$\|F^{(m)}(u_1, \ldots, u_m)\| \lesssim \|u_1\| \cdots \|u_m\|, \quad \|G^{(m)}(u_1, \ldots, u_m)\| \lesssim \|u_1\| \cdots \|u_m\|.$$  

(4.9)

The hidden constants can be chosen independent of $x_0$ but dependent on the compact set $K$.

By Taylor’s theorem,

$$x_h + f(x_h, \sum_\ell w_\ell g(x_h, x_{h+\ell} - x_h)) = x_h + \sum_{m=1}^k F^{(m)}(\omega_1, \ldots, \omega) + O(\|\omega\|^{k+1}),$$

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where
\[ \omega := \omega_h := \left( x_h - x_0, \sum_\ell w_\ell g(x_h, x_{h+\ell} - x_h) \right). \]

Using \( \sum_\ell w_\ell = 1 \), \( \omega \) can be rewritten as \( \omega = \sum_\ell w_\ell \kappa_\ell \), where
\[ \kappa_\ell := \kappa_{\ell,h} := \left( x_h - x_0, g(x_h, x_{h+\ell} - x_h) \right). \tag{4.10} \]

Since \( K \) is compact and \( g \) is smooth, in particular Lipschitz, there is a constant \( A > 0 \) such that
\[ \|g(x, y_1) - g(x, y_2)\| \leq A \|y_1 - y_2\|, \quad \forall x \in K, \; y_1, y_2 \in B(0, r). \tag{4.11} \]

Therefore, \( \max_{\ell, h} \|\kappa_{\ell,h}\| = O(\|\Delta x\|) \) and \( \max_\ell \|\omega_\ell\| = O(\|\Delta x\|) \). Here \( \|\Delta x\| \) simply means \( \max_\ell \|x_{\ell+1} - x_\ell\| \).

This, together with the multi-linearity of \( F^{(m)} \), yields:
\[ f(x_h, \sum_\ell w_\ell g(x_h, x_{h+\ell} - x_h)) = \sum_{m=1}^k \sum_{\ell_1, \ldots, \ell_m} w_{\ell_1} \cdots w_{\ell_m} F^{(m)}(\kappa_{\ell_1}, \ldots, \kappa_{\ell_m}) + O(\|\Delta x\|^{k+1}). \tag{4.12} \]

On the other hand,
\[ \sum_\ell w_\ell x_{h+\ell} = x_h + \sum_\ell w_\ell f(x_h, g(x_h, x_{h+\ell} - x_h)) \]
\[ = x_h + \sum_\ell w_\ell \sum_{m=1}^k F^{(m)}(\kappa_{\ell_1}, \ldots, \kappa_{\ell_m}) + O(\|\Delta x\|^{k+1}) \tag{4.13} \]
\[ = x_h + \sum_{m=1}^k \sum_\ell w_\ell F^{(m)}(\kappa_{\ell_1}, \ldots, \kappa_{\ell_m}) + O(\|\Delta x\|^{k+1}). \]

Assume \( \mathcal{N}(0, 0) = \{-L, \ldots, -L + p\} \). Recall from the definition of \( \mathcal{N}(i, 0) \) that \( p \) is a finite number that only depends on the finitely supported \((a_\ell), (b_\ell)\) and \((c_\ell)\). Moreover, since the polynomial reproduction orders of \( T \) and \( T \) are at least \( k \), \( p \geq k \).

For \( \ell \geq 0 \), let
\[ D_\ell := \text{the } \ell\text{-th order difference of } x_{-L}, \ldots, x_{-L+\ell} = \sum_{j=0}^\ell (-1)^{\ell-j} \binom{\ell}{j} x_{-L+j}. \tag{4.14} \]

We can recover \( x_{-L+\ell} \) from \( D_0, \ldots, D_{\ell} \) by
\[ x_{-L+\ell} = \sum_{j=0}^\ell \binom{\ell}{j} D_j. \tag{4.15} \]

For any \( -L \leq \ell \leq -L + p \),
\[ x_\ell - x_0 = \sum_{j=1}^p A_\ell^j D_j, \tag{4.16} \]
where
\[ A_\ell^j := \binom{L+\ell}{j} - \binom{L}{j}. \tag{4.17} \]

Note that
\[ A_\ell^* := \frac{(L+\bullet)(L+\bullet+1) \cdots (L+\bullet+j-1)}{j!} - \frac{L(L-1) \cdots (L-j+1)}{j!} \in \Pi_j, \tag{4.18} \]
and \( A_\ell^* \) is the evaluation of this polynomial at \( \ell \), and \( A_\ell^j \) can be thought of as a polynomial function in the variable \( x \). In the sequel, a lot of polynomials will be formed out of these polynomials.
Since $p \geq k$, we can also rewrite (4.16) as
\[ x_\ell - x_0 = \sum_{j=1}^{k} A_j^h D_j + O(\Omega_k(x_\ell)) \] (4.19)

3° The first component of $\kappa_{\ell,h}$ is exactly (4.19); the second component of $\kappa_{\ell,h}$ can be written as
\[ g(x_h, x_{h+\ell} - x_h) = \sum_{n=1}^{k} G^{(n)}(\eta_{\ell,h}, \ldots, \eta_{\ell,h}) + O(\|\Delta x\|^{k+1}), \]

where
\[ \eta_{\ell,h} = (x_h - x_0, x_{h+\ell} - x_h) \]
\[ = \left( \sum_{j=1}^{k} A_j^h D_j, \sum_{j=1}^{k} (A_j^{h+\ell} - A_j^h) D_j \right) + O(\Omega_k(x_\ell)) \]
\[ = \sum_{j=1}^{k} A_j^h (D_j, 0) + (A_j^{h+\ell} - A_j^h)(0, D_j) + O(\Omega_k(x_\ell)). \]

If we define
\[ D'_j := \begin{cases} (D_j, 0), & \epsilon = 0 \\ (0, D_j), & \epsilon = 1 \end{cases} \] (4.20)

then
\[ \eta_{\ell,h} = \sum_{\epsilon=0}^{1} \sum_{j=1}^{k} (A_j^{h+\ell} - \epsilon A_j^h) D'_j + O(\Omega_k(x_\ell)). \] (4.21)

We introduce the shorthand notations
\[ \mathcal{E}_k := \{ I = (\epsilon_1, \ldots, \epsilon_n) : 1 \leq n \leq k, \epsilon_i = 0, 1 \}, \]
\[ \mathcal{J}_k := \{ J = (j_1, \ldots, j_n) : 1 \leq n \leq k, 1 \leq j_i \leq k \}. \]

For $I \in \mathcal{E}_k, |I| := n$, $|I| := \epsilon_1 + \cdots + \epsilon_n$. Similarly, for $J \in \mathcal{J}_k, |J| := n$, $|J| := j_1 + \cdots + j_n$. Define also
\[ \mathcal{E}_k \otimes \mathcal{J}_k := \{ (I, J) \in \mathcal{E}_k \times \mathcal{J}_k : |I| = |J| \} \] (4.22)

By multi-linearity of $G^{(n)}$ and also (4.9),
\[ G^{(n)}(\eta_{\ell,h}, \ldots, \eta_{\ell,h}) = \sum_{I \in \mathcal{E}_k} \sum_{J \in \mathcal{J}_k} Q_{I,J}^{h,\ell} G^{(n)}(D_{I,J}) + O(\Omega_k(x_\ell)), \]

where $D_{I,J} := (D_{j_1}^{i_1}, \ldots, D_{j_n}^{i_n})$ and $Q_{I,J}^{h,\ell} := \prod_{i=1}^{n} (A_{j_i}^{h+\epsilon_i} - \epsilon_i A_{j_i}^h)$. By the comment below (4.18), we can view, for any fixed $I = (\epsilon_1, \ldots, \epsilon_n)$ and $J = (j_1, \ldots, j_n)$,
\[ Q_{I,J}^{x,y} := \prod_{i=1}^{n} (A_{j_i}^{x+\epsilon_i y} - \epsilon_i A_{j_i}^x) \] (4.23)

as a bivariate polynomial in $x$ and $y$ of total degree $|J| = j_1 + \cdots + j_n$.

So we have
\[ \kappa_{\ell,h} = \left( \sum_{j=1}^{k} A_j^h D_j, \sum_{n=1}^{k} \sum_{(I,J) \in \mathcal{E}_k \otimes \mathcal{J}_k} Q_{I,J}^{h,\ell} G^{(n)}(D_{I,J}) \right) + O(\|\Delta x\|^{k+1}) + O(\Omega_k(x_\ell)) = O(\Omega_k(x_\ell)). \]
To simplify this expression, note that $A_j^h$ is actually the $Q_{j,l}^{h,l}$ with $J = (j), \ I = (0)$ ([J] = [I] = 1). Therefore, $\kappa_{\ell,h}$ can be written as

$$\kappa_{\ell,h} = \sum_{(j,l) \in E_k \otimes J_k} Q_{j,l}^{h,l} U_j^{\ell} + O(\Omega_k(x_\ell)), \quad (4.24)$$

where

$$U_j^{\ell} := \begin{cases} 
(0, G^{(n)}(D_j^{\ell})) & \text{if } [I] = [J] = n > 1, \text{ or } I = (1), J = (j) \\
(D_j, G^{(1)}(D_j^{(0)})) & \text{if } I = (0), J = (j)
\end{cases}. \quad (4.25)$$

4° We now combine (4.12) with (4.24) and (4.9).

$$f(x_h, \sum_\ell w_\ell g(x_h, x_{h+\ell} - x_h))$$

$$= \sum_{m=1}^k \sum_{\ell_1, \ldots, \ell_m} w_{\ell_1} \cdots w_{\ell_m} \sum_{(i,j) \in E_k \otimes J_k} Q_{i,j}^{h,\ell_1} \cdots Q_{j,m}^{h,\ell_m} F^{(m)}(U_{j,1}^{\ell_1}, \ldots, U_{j,m}^{\ell_m}) + O(\Omega_k(x_\ell))$$

$$= \sum_{m=1}^k \sum_{i=1}^k \sum_{\ell_1, \ldots, \ell_m} w_{\ell_1} \cdots w_{\ell_m} Q_{i,1}^{h,\ell_1} \cdots Q_{j,m}^{h,\ell_m} F^{(m)}(U_{j,1}^{\ell_1}, \ldots, U_{j,m}^{\ell_m}) + O(\Omega_k(x_\ell)) \quad (4.26)$$

$$= \sum_{m=1}^k \sum_{i=1}^k \sum_{\ell_1, \ldots, \ell_m} w_{\ell_1} \cdots w_{\ell_m} Q_{i,1}^{h,\ell_1} \cdots Q_{j,m}^{h,\ell_m} F^{(m)}(U_{j,1}^{\ell_1}, \ldots, U_{j,m}^{\ell_m}) + O(\Omega_k(x_\ell)).$$

Next, we combine (4.13) with (4.24) and (4.9).

$$\sum_\ell w_\ell x_{h+\ell} = x_h + \sum_{m=1}^k \sum_\ell w_\ell F^{(m)}(\kappa_{\ell,h}, \ldots, \kappa_{\ell,h}) + O(\|\Delta x\|^{k+1})$$

$$= x_h + \sum_{m=1}^k \sum_{i=1}^m \sum_{\ell} w_\ell Q_{i,1}^{h,\ell} \cdots Q_{j,m}^{h,\ell} F^{(m)}(U_{j,1}^{\ell_1}, \ldots, U_{j,m}^{\ell_m}) + O(\Omega_k(x_\ell)) \quad (4.27)$$

$$= x_h + \sum_{m=1}^k \sum_{i=1}^m \sum_{\ell} w_\ell \prod_{i=1}^m Q_{i,1}^{h,\ell} F^{(m)}(U_{j,1}^{\ell_1}, \ldots, U_{j,m}^{\ell_m}) + O(\Omega_k(x_\ell)).$$

By (4.26) and (4.27), we have now quantified the difference between $x_h + f(x_h, \sum_\ell w_\ell g(x_h, x_{h+\ell} - x_h))$ and $\sum_\ell w_\ell x_{h+\ell}$

$$x_h + f(x_h, \sum_\ell w_\ell g(x_h, x_{h+\ell} - x_h)) - \sum_\ell w_\ell x_{h+\ell}$$

$$= \sum_{m=1}^k \sum_{i=1}^m \sum_{\ell} \mathcal{Y}_{\ell}^{h,i,j_1}(I_1, \ldots, J_1, \ldots, I_m, J_m) F^{(m)}(U_{j,1}^{\ell_1}, \ldots, U_{j,m}^{\ell_m}) + O(\Omega_k(x_\ell)) \quad (4.28)$$

where

$$\mathcal{Y}_{\ell}^{h,i,j_1}(I_1, \ldots, J_1, \ldots, I_m, J_m) = \prod_{i=1}^m \sum_{\ell} w_\ell Q_{i,1}^{h,\ell} - \sum_{\ell} w_\ell \prod_{i=1}^m Q_{i,1}^{h,\ell}. \quad (4.29)$$

**Note:** The data $(x_\ell)$ does not enter the definition of $\mathcal{Y}_{\ell}^{h,i,j_1}(I_1, \ldots, J_1, \ldots, I_m, J_m)$, whereas the mask $(w_\ell)$ does not enter the definition of $F^{(m)}(U_{j,1}^{\ell_1}, \ldots, U_{j,m}^{\ell_m})$. 

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We are now ready to prove the first part of the theorem, namely (4.4). For this purpose, we use (4.28)-(4.29) with \((w_i) = (c_i)\) and we replace \((f, g)\) by \((\hat{f}, \hat{g})\) from (4.2). Since \((c_i)\) comes from the mask of a \(\Pi_k\) reproducing subdivision scheme, \(\sum_{\pi} c_{\pi}(\ell) = \pi(1/2)\) for any \(\pi \in \Pi_k\). By (4.23), \(Q_{i,j}^{h,\ell} = \pi_{p_{i,j},h}(\ell)\) for a \(\pi_{p_{i,j},h} \in \Pi_{|J|}\). Therefore, when \(|J^1| + \cdots + |J^m| \leq k\), \(\pi_{p_{i,j},h} \in \Pi_k\) and \(\prod_{i=1}^{m} \pi_{p_{i,j},h} \in \Pi_k\), so

\[
\Upsilon^h_{I^1, J^1, \ldots, I^m, J^m} = \prod_{i=1}^{m} c_{\pi} Q_{i,j}^{h,\ell} - \sum_{\ell} c_{\pi} \prod_{i=1}^{m} Q_{i,j}^{h,\ell},
\]

\[
= \prod_{i=1}^{m} (\pi_{p_{i,j},h}(1/2)) - \left( \prod_{i=1}^{m} \pi_{p_{i,j},h} \right)(1/2) = 0.
\]

By (4.30),

\[
\|(\hat{T}x)_{2h+1} - (Tx)_{2h+1}\| \leq \sum_{m=1}^{k} \sum_{i_1, \ldots, i_m} |\Upsilon^h_{I^1, J^1, \ldots, I^m, J^m}| \|F^{(m)}(U_{I^1, J^1}, \ldots, U_{I^m, J^m})\| + O(\Omega_k(x))
\]

By (4.30), the only terms that survive on the right-hand side are those with \(|J^1| + \cdots + |J^m| > k\). On the other hand,

\[
\|F^{(m)}(U_{I^1, J^1}, \ldots, U_{I^m, J^m})\| \overset{(4.9)}{\lesssim} \|U_{I^1}\| \cdots \|U_{I^m, J^m}\| \overset{(4.25)+(4.9)}{\lesssim} \|D_{J^1}\| \cdots \|D_{J_{\Sigma_{1, m}}^m}\|
\]

where \((J_1, \ldots, J_{\Sigma_{1, m}})\) is the concatenation of \(J^1, \ldots, J^m\). By recalling the definition of \(\Omega_k((x_i)_{\ell})\) and \(D_j\), we finish the proof of (4.4).

By (4.28)-(4.29),

\[
(Sx)_{2h} - (Tx)_{2h} = \sum_{m=1}^{k} \sum_{i_1, \ldots, i_m} \Upsilon^h_{I^1, J^1, \ldots, I^m, J^m} F^{(m)}(U_{I^1, J^1}, \ldots, U_{I^m, J^m}) + O(\Omega_k(x)),
\]

where \(\Upsilon^h_{I^1, J^1, \ldots, I^m, J^m} = \prod_{i=1}^{m} a_{\ell} Q_{I^1, J^1}^{h,\ell} - \sum_{\ell} a_{\ell} \prod_{i=1}^{m} Q_{i,j}^{h,\ell}\).

We now show that

\[
\Upsilon_{I^1, J^1, \ldots, I^m, J^m} := \prod_{i=1}^{m} a_{\ell} Q_{I^1, J^1}^{h,\ell} - \sum_{\ell} a_{\ell} \prod_{i=1}^{m} Q_{I^1, J^1}^{h,\ell} \in \Pi_{|J^1| + \cdots + |J^m| - 2}.
\]

If \(J = (j_1, \ldots, j_n)\) and \(I = (\epsilon_1, \ldots, \epsilon_n)\), by (4.18) and (4.23),

\[
Q_{J,I}^{y} = \frac{1}{j_1! \cdots j_n!} x_{|J|} + B_{J,I}(y)x_{|J|-1} + \sum_{i=|J|-1} B_{J,I}(y)x^i.
\]

where \(B_{J,I}(y)\) and \(B_{J,I}(y)\) are polynomials of \(y\) dependent on \(I, J\). From this,

\[
\sum_{\ell} a_{\ell} Q_{J,I}^{y,\ell} = \frac{1}{j_1!} \left( x_{|J|} + \sum_{\ell} a_{\ell} B_{I,J}(\ell) \cdot x_{|J|-1} + \cdots \right).
\]

In above and below, \(J! := j_1! \cdots j_n!\) and the \(\cdots\) involves terms of \(x^i\) with \(i < |J| - 1\). Therefore

\[
\prod_{i=1}^{m} a_{\ell} Q_{I^1, J^1}^{y,\ell} = \frac{1}{j_1! \cdots j_m!} \left( x_{|J^1| + \cdots + |J^m|} + \sum_{i=1}^{m} a_{\ell} B_{I^1, J^1}(\ell) \cdot x_{|J^1| + \cdots + |J^m| - 1} + \cdots \right).
\]
On the other hand
\[ \prod_{i=1}^{m} Q_{i,j}^{x,\ell} = \frac{1}{j! \cdots m!} \left( x^{\left| j^{1} \right| + \cdots + \left| j^{m} \right|} + \sum_{i=1}^{m} B_{i,j}(\ell), x^{\left| j^{1} \right| + \cdots + \left| j^{m} \right| - 1} + \ldots \right), \]

so
\[ \sum_{\ell} \alpha_{\ell} \prod_{i=1}^{m} Q_{i,j}^{x,\ell} = \frac{1}{j! \cdots m!} \left( x^{\left| j^{1} \right| + \cdots + \left| j^{m} \right|} + \sum_{\ell} \alpha_{\ell} \sum_{i=1}^{m} B_{i,j}(\ell), x^{\left| j^{1} \right| + \cdots + \left| j^{m} \right| - 1} + \ldots \right). \]

And the claim (4.33) is proved.

7° Since we have established (4.4) in 5°, we can now write
\[ x_{h+1/2} = x_{h+1/2} = \sum_{\ell} c_{\ell} x_{h+\ell} + O(\Omega_{k}((x_{\ell}))) = x_{0} + \sum_{\ell} c_{\ell} (x_{h+\ell} - x_{0}) + O(\Omega_{k}((x_{\ell}))) \]
\[ = x_{0} + \sum_{\ell} c_{\ell} \sum_{j=1}^{p} A_{j}^{h+\ell} D_{j} + O(\Omega_{k}((x_{\ell}))) \]
\[ = x_{0} + \sum_{\ell} \left( \sum_{j=1}^{p} c_{\ell} A_{j}^{h+\ell} \right) D_{j} + O(\Omega_{k}((x_{\ell}))) \]
\[ = x_{0} + \sum_{j=1}^{k} \left( \sum_{\ell} c_{\ell} A_{j}^{h+\ell} \right) D_{j} + \sum_{j>k} \left( \sum_{\ell} c_{\ell} A_{j}^{h+\ell} \right) D_{j} + O(\Omega_{k}((x_{\ell}))) \]
\[ = x_{0} + \sum_{j=1}^{k} A_{j}^{h+1/2} D_{j} + O(\Omega_{k}(x_{\ell})). \]

In the last equality, \( \sum_{j>k} \left( \sum_{\ell} c_{\ell} A_{j}^{h+\ell} \right) D_{j} \) can be absorbed into \( O(\Omega_{k}((x_{\ell}))) \) because \( \| D_{j} \|_{\infty} \leq |\Delta^{i}(x_{\ell})|_{\infty} \) and \( |\Delta^{j+1}(x_{\ell})|_{\infty} \leq 2 |\Delta^{j}(x_{\ell})|_{\infty} \).

8° In this step, we basically need to repeat most of the derivations in 1°, 2°, 3° and 4° but for \((Sx)_{2h+1} - (Tx)_{2h+1}\) instead of \((Sx)_{2h} - (Tx)_{2h}\). While the odd rule of \( S \) involves a change in the “base point” from \( x_{h} \) to \( x_{h+1/2} \), we shall continue to use the Taylor expansions of \( f \) and \( g \) at the same base points (4.8). In the process, we shall use the approximation of \( x_{h+1/2} \) derived in (4.34).

\[ (Sx)_{2h+1} - (Tx)_{2h+1} = f(x_{h+1/2}, x_{h+\ell} - x_{h+1/2}) - \sum_{\ell} b_{\ell} f(x_{h+1/2}, g(x_{h+1/2}, x_{h+\ell} - x_{h+1/2}) \]
\[ = \sum_{m=1}^{k} F^{(m)} \left( \sum_{\ell} b_{\ell} \tilde{k}_{\ell,h}, \ldots, \sum_{\ell} b_{\ell} \tilde{k}_{\ell,h} \right) - \sum_{\ell} b_{\ell} \sum_{m=1}^{k} F^{(m)}(\tilde{k}_{\ell,h}, \ldots, \tilde{k}_{\ell,h}) + O(\max_{\ell,h} \| \tilde{k}_{\ell,h} \|^{|k+1|}), \]

where
\[ \tilde{k}_{\ell,h} = \left( x_{h+1/2} - x_{0}, g(x_{h+1/2}, x_{h+\ell} - x_{h+1/2}) \right) \]
\[ = \left( x_{h+1/2} - x_{0}, \sum_{n=1}^{k} G^{(n)}(\tilde{y}_{h,\ell}, \ldots, \tilde{y}_{h,\ell}) + O(\max_{\ell,h} \| \tilde{y}_{h,\ell} \|^{|k+1|}), \right) \]
\[
\tilde{\eta}_{\ell, h} = (x_{h+\frac{1}{2}} - x_0, x_{h+\frac{1}{2}} - x_{h+1/2})
\]

\[
= \left( \sum_{j=1}^{k} A_j^{h+1/2} D_j, \sum_{j=1}^{k} (A_j^{h+\ell} - A_j^{h+1/2}) D_j \right) + O(\Omega_k(x_\ell)) \tag{4.37}
\]

\[
= \left( \sum_{j=1}^{k} A_j^{h+1/2} D_j, \sum_{j=1}^{k} (A_j^{h+1/2} - \epsilon A_j^{h+1/2}) D_j \right) + O(\Omega_k(x_\ell)). \tag{4.38}
\]

The trick in the last equality, as we will see, will be crucial; also it is worth comparing (4.37) with (4.21).

We can now write

\[
\max_{\ell, h} \|\tilde{\eta}_{\ell, h}\|^{k+1} = O(\Omega_k(x_\ell)), \quad \max_{\ell, h} \|\tilde{\eta}_{\ell, h}\|^{k+1} = O(\Omega_k(x_\ell)),
\]

and

\[
\tilde{\eta}_{\ell, h} = \left( \sum_{j=1}^{k} A_j^{h+1/2} D_j, \sum_{n=1}^{G(n)} (\tilde{\eta}_{\ell, h}, \ldots, \tilde{\eta}_{\ell, h}) \right) + O(\Omega_k(x_\ell))
\]

\[
= \left( \sum_{j=1}^{k} A_j^{h+1/2} D_j, \sum_{n=1}^{G(n)} (\tilde{\eta}_{\ell, h}, \ldots, \tilde{\eta}_{\ell, h}) \right) + O(\Omega_k(x_\ell)) \tag{4.38}
\]

\[
= \left( \sum_{j=1}^{k} A_j^{h+1/2} D_j, \sum_{n=1}^{G(n)} (\tilde{\eta}_{\ell, h}, \ldots, \tilde{\eta}_{\ell, h}) \right) + O(\Omega_k(x_\ell)) \tag{4.38}
\]

\[
= \sum_{(I, J) \in E_k \otimes J_k} \tilde{Q}^{h+1/2, \ell-1/2}_I + O(\Omega_k(x_\ell)).
\]

In the second last equality, we used the notation \(D_{I, J}\) and \(\tilde{Q}^{h, y}_{I, J}\) defined at and around (4.23), we also used the observation made in the last step of (4.37); in the last equality, we used the \(U^j_I\) defined in (4.25). Compare (4.38) with (4.24).

Similar to how we got (4.32), by combining (4.35) with (4.38), we have

\[
(Sx)_{2h+1} - (Tx)_{2h+1} = \sum_{m=1}^{k} \sum_{(I, J) \in E_k \otimes J_k} \tilde{z}^h_{(I, J), \ldots, (I, J)} F^{(m)}(U^j_I, \ldots, U^m_I) + O(\Omega_k(x_\ell)), \tag{4.39}
\]

where

\[
\tilde{z}^h_{(I, J), \ldots, (I, J)} = \prod_{\ell=1}^{m} \sum_{I} b_I \tilde{Q}^{h+1/2, \ell-1/2}_I - \sum_{I} b_I \prod_{\ell=1}^{m} \tilde{Q}^{h+1/2, \ell-1/2}_I.
\]

9° If we define the sequence \(\tilde{\chi}_{(I, J), \ldots, (I, J)}\) by

\[
(\tilde{\chi}_{(I, J), \ldots, (I, J)})_{2h} = \tilde{Y}^h_{(I, J), \ldots, (I, J)}, \quad (\tilde{\chi}_{(I, J), \ldots, (I, J)})_{2h+1} = \tilde{z}^h_{(I, J), \ldots, (I, J)},
\]

then we can write

\[
\left( \Delta^{k-1} Sx - \Delta^{k-1} Tx \right)_{2h} = \sum_{m=1}^{k} \sum_{(I, J) \in E_k \otimes J_k} \tilde{\chi}_{(I, J), \ldots, (I, J)} F^{(m)}(U^j_I, \ldots, U^m_I) + O(\Omega_k(x_\ell)),
\]

\[
\left( \Delta^{k-1} Sx - \Delta^{k-1} Tx \right)_{2h+1} = \sum_{m=1}^{k} \sum_{(I, J) \in E_k \otimes J_k} \tilde{\chi}_{(I, J), \ldots, (I, J)} F^{(m)}(U^j_I, \ldots, U^m_I) + O(\Omega_k(x_\ell)).
\]

(4.40)
By (4.7) and (4.23), when $|J^1| + \cdots + |J^m| \leq k$,

$$
\Xi_{(I^1,J^1),\ldots,(I^m,J^m)}^h = \prod_{i=1}^m \sum_{\ell} b_{\ell} Q_{I^1,J^1}^{h+1/2,\ell-1/2} - \sum_{\ell} a_{\ell} \prod_{i=1}^m Q_{I^1,J^1}^{h+1/2,\ell} = \prod_{i=1}^m a_{\ell} Q_{I^1,J^1}^{h+1/2,\ell-1/2}.
$$

(4.41)

Combining this with (4.33), when $|J^1| + \cdots + |J^m| \leq k$, the sequence $\chi_{(I^1,J^1),\ldots,(I^m,J^m)}$ is a polynomial of degree $\leq |J^1| + \cdots + |J^m| - 2$ sampled on a uniform grid, and hence its $(k-1)$-th differences must vanish. In other words, all the terms in (4.40) with $|J^1| + \cdots + |J^m| \leq k$ vanish, and (4.5) is proved once we apply an argument akin to the one in (4.31).

Combining Theorem 2.4 and Theorem 4.1, we know that if we begin with a dense enough initial sequence, then locally all the subdivision points are well-defined and stay within a single chart; moreover the smoothness of the limit curve is as high as that of the underlying linear subdivision scheme. In other words, we have the following smoothness equivalence theorem.

**Theorem 4.2.** For any $C^k$ interpolatory linear subdivision scheme $\hat{T}$, if $T$ is $C^k$, the corresponding nonlinear interpolatory subdivision scheme $\hat{S}$, defined by (3.11), is also $C^k$. For any $C^k$ stable linear subdivision scheme $T$, if $T$ is $C^k$, the corresponding nonlinear subdivision scheme $S$, defined by (3.9), is also $C^k$.

**Remark.** From the linear theory, if $\hat{T}$ is $C^k$, then $T$ must reproduce $\Pi_k$ (i.e. condition (4.6).) Similarly, if $T$ is $C^k$, then $T$ must reproduce $\Pi_k$ (i.e. condition (4.7).) Of course $\hat{T}$ (resp. $T$) has to be $C^k$ smooth before we can expect $\hat{S}$ (resp. $S$) to be $C^k$ smooth. However, recall that the construction of $S$ uses $\hat{T}$, $\hat{T}$ needs not be $C^k$ smooth, it only needs to reproduce $\Pi_k$ in order for Theorem 4.2 to go through.

## A Appendix

### A.1 Proof of Lemma 2.1

(i) We prove that for each $J \in \mathbb{N}$, there exists a $\delta_J > 0$ such that $S^J x$ is well-defined for any $x \in \mathcal{X}_{M,\delta}$. It is equivalent to showing that there exists a $\delta_J > 0$ such that $|\Delta S^J x|_\infty < \delta$ for $j = 0, \cdots, J-1$ and $x \in \mathcal{X}_{M,\delta}$.

We use induction. When $J = 1$, we can choose $\delta_J = \delta$. Suppose $|\Delta S^J x|_\infty < \delta$ for $j = 0, \cdots, J-1$ and $x \in \mathcal{X}_{M,\delta}$.

Since $\hat{S}$ is convergent, there exist $\alpha > 0$ and $C > 0$ such that

$$
|\Delta S^J x|_\infty \leq C D^{-\alpha} |\Delta x|_\infty
$$

for any $x \in \mathcal{X}_M$ and $j \in \mathbb{N}$.

It follows that

$$
|\Delta S^J x|_\infty \leq |\Delta S S^{J-1} x|_\infty + |\Delta S^J x - \Delta S S^{J-1} x|_\infty \\
\leq CD^{-\alpha} |\Delta S^{J-1} x|_\infty + 2|S^J x - S S^{J-1} x|_\infty \\
\leq CD^{-\alpha} |\Delta S^{J-1} x|_\infty + 2C_1 |\Delta S^{J-1} x|_\infty \\
\leq (CD^{-\alpha} + 2C_1 \delta)|\Delta S^{J-1} x|_\infty \\
\leq \cdots \\
\leq (CD^{-\alpha} + 2C_1 \delta)^J |\Delta x|_\infty.
$$

(4.1)

Therefore we can choose

$$
\delta_{J+1} := \min \left( \delta_J, \frac{\delta}{(CD^{-\alpha} + 2C_1 \delta)^J} \right).
$$

\footnote{The ‘M’ in Theorem 2.4 plays the role of a chart in the manifold setting.}
(ii) We prove that for each $K \in \mathbb{N}$, there exists a $C_K > 0$ such that
\[ |S^K x - S^K x|_\infty \leq C_K \Omega_1(x) \]
whenever $S^K x$ is well-defined.

We use induction. The statement is true for $K = 1$ by assumption. Suppose that it is true for some $K \in \mathbb{N}$. Then it follows from (A.1) that
\[ |S^{K+1} x - S^{K+1} x|_\infty \leq |S^{K+1} x - S S^K x|_\infty + |S S^K x - S^{K+1} x|_\infty \]
\[ \leq C_1 |\Delta S^K x|_\infty + |\Delta S^K x|_\infty C_K |\Delta x|_\infty \]
\[ \leq C_1 (\bar{C}D^{-\alpha} + 2C_1 \delta)^{2K} |\Delta x|_\infty^2 + |\bar{S}|_\infty C_K |\Delta x|_\infty^2 \]
\[ = (C_1 (\bar{C}D^{-\alpha} + 2C_1 \delta)^{2K} + |\bar{S}|_\infty C_K) |\Delta x|_\infty^2. \]

Therefore we can choose $C_{K+1} := C_1 (\bar{C}D^{-\alpha} + 2C_1 \delta)^{2K} + |\bar{S}|_\infty C_K$.

(iii) We prove that there exist $C > 0$, $\beta > 0$ and $\delta_0 > 0$ such that
\[ |\Delta S^J x|_\infty \leq CD^{-\beta} |\Delta x|_\infty \]
whenever $S^J x$ is well-defined and $|\Delta x|_\infty < \delta_0$.

Choose any $0 < \beta < \alpha$ and $K \in \mathbb{N}$ such that $K > \frac{\log \beta C}{\alpha - \beta}$. Then $\bar{C}D^{-K\alpha} < D^{-K\beta}$. It follows from (i)(ii) that there exists $C_K > 0$ and $\delta_K > 0$ such that $S^K x$ exists for $x \in \mathcal{X}_{M,\delta_K}$ and
\[ |S^K x - S^K x|_\infty \leq C_K |\Delta x|_\infty^2. \]

Let $\delta'_K := \min(\delta_K, \frac{D^{-K\beta} \bar{C} D^{-K\alpha}}{2C_0})$. Then for any $x \in \mathcal{X}_{M,\delta'_K}$, we have
\[ |\Delta S^K x|_\infty \leq |\Delta S^K x|_\infty + |\Delta S^K x - \Delta S^K x|_\infty \]
\[ \leq \bar{C}D^{-K\alpha} |\Delta x|_\infty + 2C_0 |\Delta x|_\infty^2 \]
\[ = (\bar{C}D^{-K\alpha} + 2C_0 |\Delta x|_\infty) |\Delta x|_\infty \]
\[ \leq D^{-K\beta} |\Delta x|_\infty. \quad \text{(A.2)} \]

It follows from (A.1) that whenever $S^J x$ is well-defined, we have
\[ |\Delta S^J x|_\infty \leq \bar{C}' |\Delta x|_\infty, \quad \text{(A.3)} \]

where $\bar{C}' = \max(\bar{C}D^{-\alpha} + 2C_1 \delta, 1)$.

Let $\delta_0 := \bar{C}' \delta'_K$. Then it follows from (A.2)(A.3) that whenever $S^J x$ is well-defined and $|\Delta x|_\infty < \delta_0$,
\[ |\Delta S^J x|_\infty = |\Delta S^{tK} x|_\infty \leq D^{-tK\beta} |\Delta S^r x|_\infty \leq D^{-tK\beta} \bar{C}' |\Delta x|_\infty = D^{-t\beta} D^{t\beta} \bar{C}' |\Delta x|_\infty \leq CD^{-\beta} |\Delta x|_\infty, \]

where $C := \max_{r < k} D^{t\beta} \bar{C}'$.

(iv) Last, we prove that there exists a $\delta' > 0$ such that $S^J x$ is well-defined for any $j \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$.

Let $\delta' := \min(\delta, \delta/C, \delta_0)$ where $C$ and $\delta_0$ are from (iii). Then we use induction to prove that $S^J x$ is well-defined for any $j \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$. When $j = 1$, this is obviously true. Suppose $S^J x$ is well-defined for any $x \in \mathcal{X}_{M,\delta'}$. Then it follows from (iii) that
\[ |\Delta S^J x|_\infty \leq CD^{-\beta} |\Delta x|_\infty < C |\Delta x|_\infty \leq \delta, \]

for any $x \in \mathcal{X}_{M,\delta'}$. Therefore $S^{J+1} x$ is well-defined for any $x \in \mathcal{X}_{M,\delta'}$.

Combining (iii) and (iv), we have
\[ |\Delta S^J x|_\infty \leq CD^{-\beta} |\Delta x|_\infty \]
for any $j \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$. \qed
A.2 Auxiliary lemmas for Theorem 2.4

Lemma A.1. Let $S$ be a subdivision operator defined on $X_{M,\delta}$ and $\bar{S}$ be a linear subdivision operator. Both of them have the same dilation factor $D$. Suppose there exist $\mu_1, \ldots, \mu_{k+1} \in [0, 1)$ such that
\[
|\Delta_j \bar{S} x|_\infty \leq D^{-j+\mu_j} |\Delta_j x|_\infty, \quad \forall x, j = 1, \ldots, k + 1,
\]
and
\[
\mu_j < \frac{\mu_{j+1}}{j+1}, \quad j = 1, \ldots, k.
\]
If there exists $C > 0$ such that for any $x \in X_{M,\delta}$ and $j = 1, \ldots, k$
\[
|\Delta_j \bar{S} x - \Delta_j^{-1} \bar{S} x|_\infty \leq C \Omega_j(x).
\]
Then for any $0 < \epsilon < \min_{1 \leq j \leq k} \left(\frac{\mu_{j+1}}{j+1} - \mu_j\right)$, there exist $0 < \delta' \leq \delta$ and polynomials $P_2, \ldots, P_{k+1}$ such that for any $n \in \mathbb{N}$ and any $x \in X_{M,\delta'}$
\[
|\Delta^n \bar{S} x|_\infty \leq D^{(-1+\mu_1+\epsilon)n} |\Delta x|_\infty,
\]
\[
|\Delta^n \bar{S} x|_\infty \leq D^{(-j+\mu_j)n} P_j(n) |\Delta x|_\infty, \quad j = 2, \ldots, k + 1.
\]

Proof: Since $\Omega_1(x) = |\Delta_1 x|_\infty$, the case has been proved in [13, Theorem 2], i.e. for any $0 < \epsilon < \min_{1 \leq j \leq k} \left(\frac{\mu_{j+1}}{j+1} - \mu_j\right)$, there exists $0 < \delta' \leq \delta$ such that for any $x \in X_{M,\delta'}$ and any $n \in \mathbb{N}$,
\[
|\Delta^n \bar{S} x|_\infty \leq D^{(-1+\mu_1+\epsilon)n} |\Delta x|_\infty.
\]
We proceed by induction in $j$. Now assume the result is true for $1, \ldots, j \leq k$. Now for any $n \in \mathbb{N}$,
\[
|\Delta^n \bar{S} x|_\infty \leq |\Delta^n \bar{S} S^{n-1} x|_\infty + |\Delta^n \bar{S} S^n x - \Delta^n \bar{S} S^{n-1} x|_\infty.
\]
Since for any sequences $y, z$, we have $|\Delta y - \Delta z|_\infty = |\Delta (y - z)|_\infty \leq 2|y - z|_\infty$. It follows that
\[
|\Delta^n \bar{S} x|_\infty \leq |\Delta^n \bar{S} S^{n-1} x|_\infty + 4|\Delta^n \bar{S} S^{n-1} x|_\infty
\]
\[
\leq D^{(-1+\mu_1+\epsilon)n} |\Delta^n \bar{S} S^{n-1} x|_\infty + 4C \Omega_j(S^n x).
\]
Denote $d_n := |\Delta^n \bar{S} S^n x|_\infty$. Then
\[
d_n \leq D^{(-j+\mu_j+1)n} d_{n-1} + 4C \Omega_j(S^{n-1} x)
\]
\[
\leq \cdots
\]
\[
\leq D^{(-j+\mu_j+1)n} d_0 + 4C \sum_{i=0}^{n-1} D^{(\mu_j+1) + (n-i) \Omega_j(S^i x)}
\]
\[
\leq D^{(-j+\mu_j+1)n} \left( d_0 + 4C D^{(\mu_j+1) + (n-1) \Omega_j(S^n x)} \right).
\]
Since
\[
\Omega_j(S^i x) = \sum_{\gamma \in \Gamma_j} |\Delta^j S^\gamma x|_\infty^\gamma \cdots |\Delta^j S^\gamma x|_\infty^\gamma
\]
\[
\leq \sum_{\gamma \in \Gamma_j} \left( (D^{-(1+\mu_1+\epsilon)j})^\gamma \cdots (D^{-(1+\mu_j+\epsilon)j})^\gamma \right)
\]
\[
= \sum_{\gamma \in \Gamma_j} |\Delta^\gamma x|_\infty^\gamma P_2(i)^\gamma \cdots P_j(i)^\gamma D^{(\mu_1 \gamma_1 + \cdots + \mu_j \gamma_j + \epsilon \gamma_j + 1)}
\]
\[
\leq \sum_{\gamma \in \Gamma_j} |\Delta^\gamma x|_\infty^\gamma P_2(i)^\gamma \cdots P_j(i)^\gamma D^{(\mu_j \gamma_j + 1)}
\]
\[
\leq \sum_{\gamma \in \Gamma_j} |\Delta^\gamma x|_\infty^\gamma P_2(i)^\gamma \cdots P_j(i)^\gamma D^{(\mu_j \gamma_j + 1) (\mu_j + \epsilon - 1)}
\]
\[
\leq |\Delta x|_\infty \sum_{\gamma \in \Gamma_j} \delta^{(\gamma_j - 1)} P_2(i)^\gamma \cdots P_j(i)^\gamma D^{(j+1) (\mu_j + \epsilon - 1)}
\]
It follows from Theorem 2.3 that

\[ d_0 = |\Delta^{j+1} x|_\infty \leq 2^j |\Delta x|_\infty. \]

It follows that

\[
\begin{align*}
 d_n & \leq D^{(-j-1 + \mu_j + 1)n} \left( 2^j + 4CD^{j+1} \sum_{i=0}^{n-1} \sum_{\gamma \in \Gamma_j} \delta^{\gamma(i)-1} P_2(i)^{\gamma_1} \cdots P_j(i)^{\gamma_j} D^{(j+1)(\mu_j + \epsilon - \frac{\mu_j + 1}{j+1})} \right) |\Delta x|_\infty \\
 & \leq D^{(-j-1 + \mu_j + 1)n} \left( 2^j + 4CD^{j+1} \sum_{i=0}^{n-1} \sum_{\gamma \in \Gamma_j} \delta^{\gamma(i)-1} P_2(i)^{\gamma_1} \cdots P_j(i)^{\gamma_j} D^{(j+1)(\mu_j + \epsilon - \frac{\mu_j + 1}{j+1})} \right) |\Delta x|_\infty \\
 & \leq D^{(-j-1 + \mu_j + 1)n} \left( 2^j + 4CD^{j+1} \sum_{i=0}^{n-1} \sum_{\gamma \in \Gamma_j} \delta^{\gamma(i)-1} P_2(i)^{\gamma_1} \cdots P_j(i)^{\gamma_j} \right) |\Delta x|_\infty,
\end{align*}
\]

where the last inequality is due to the fact that \( \mu_j + \epsilon - \frac{\mu_j + 1}{j+1} < 0 \). Define

\[ P_{j+1}(n) = 2^j + 4CD^{j+1} \sum_{i=0}^{n-1} \sum_{\gamma \in \Gamma_j} \delta^{\gamma(i)-1} P_2(i)^{\gamma_1} \cdots P_j(i)^{\gamma_j}. \]

Then

\[ |\Delta^{j+1} S^nx|_\infty = d_n \leq D^{(-j-1 + \mu_j + 1)n} P_{j+1}(n) |\Delta x|_\infty. \]

Hence the lemma is proved. \( \square \)

**Lemma A.2.** Let \( S \) be a subdivision operator defined on \( X_{M,\delta'} \) and \( \bar{S} \) be a linear subdivision operator such that its derived subdivision operators \( S_1, \ldots, S_k \) (\( k \geq 1 \)) all exist. \( S \) and \( \bar{S} \) have the same dilation factor \( D \). Suppose there exists \( C > 0 \) such that for any \( x \in X_{M,\delta'} \) and \( j = 1, \ldots, k \)

\[ |\Delta^{j-1} Sx - \Delta^{j-1} S\bar{x}|_\infty \leq C \Omega_j(x). \]

If there exist \( \mu_1, \ldots, \mu_k+1 \in [0,1) \) satisfying

\[ \mu_j < \frac{\mu_{j+1}}{j+1}, \quad j = 1, \ldots, k, \]

and polynomials \( P_1, \ldots, P_{k+1} \) such that for any \( n \in \mathbb{N} \) and any \( x \in X_{M,\delta'} \)

\[ |\Delta^{j} S^n x|_\infty \leq D^{(-j+\mu_j)n} P_j(n) |\Delta x|_\infty, \quad j = 1, \ldots, k + 1. \]

Then \( S \) is \( C^k \).

**Proof:** It follows from Theorem 2.3 that \( S \) is continuous. We only need to show for \( j = 1, \ldots, k \), \( F^n_D(D^{jn} \Delta^{j} S^n x) \) converges uniformly as \( n \to \infty \) for all \( x \in X_{M,\delta'} \). Consider

\[
\begin{align*}
 |F_D^{n+1}(D^{jn+1})^{\Delta^{j} S^{n+1} x} - F_D^n(D^{jn} \Delta^{j} S^n x)|_\infty \\
 \leq |F_D^{n+1}(D^{jn+1})^{\Delta^{j} S^{n+1} x} - F_D^{n+1}(D^{jn+1})^{\Delta^{j} S^{n} x}|_\infty + |F_D^n(D^{jn+1})^{\Delta^{j} S^n x} - F_D^n(D^{jn} \Delta^{j} S^n x)|_\infty.
\end{align*}
\]
Using (2.2), we have
\[
|F_D^{n+1}(D^{(n+1)} \Delta^j S^{n+1} x) - F_D^{n+1}(D^{(n+1)} \Delta^j S^n x)|_\infty \\
= |D^{(n+1)} \Delta^j S^{n+1} x - D^{(n+1)} \Delta^j S^n x|_\infty \\
\leq 2D^{(n+1)}|\Delta^{-1} S^{n+1} x - \Delta^{-1} S^n x|_\infty \\
\leq 2CD^{(n+1)}\Omega_j(S^n x) \\
= 2CD^{(n+1)} \sum_{j \in \Gamma_j} |\Delta S^n x|_\infty \cdots |\Delta^j S^n x|_\infty \\
\leq 2CD^{(n+1)} \sum_{j \in \Gamma_j} \left( D^{(-1+\mu_1)n} P_1(n)|\Delta x|_\infty^\gamma \right) \cdots \left( D^{(-j+\mu_j)n} P_j(n)|\Delta x|_\infty^\gamma \right) \\
= 2CD^{(n+1)} \sum_{j \in \Gamma_j} D^n(\mu_1 \gamma_1 + \cdots + \mu_j \gamma_j - 1) P_1(n)^{\gamma_1} \cdots P_j(n)^{\gamma_j}|\Delta x|_\infty^\gamma \\
\leq 2CD^{(n+1)} \sum_{j \in \Gamma_j} D^n(\mu_j(j+1) - 1) P_1(n)^{\gamma_1} \cdots P_j(n)^{\gamma_j}|\Delta x|_\infty^\gamma \\
\leq 2CD^{(n+1)} \sum_{j \in \Gamma_j} D^n(\mu_{j+1} - 1) P_1(n)^{\gamma_1} \cdots P_j(n)^{\gamma_j}|\Delta x|_\infty^\gamma.
\]
Since \( \mu_{j+1} < 0 \) and \( P_1, \cdots, P_j \) are polynomials, it follows that
\[
\sum_{n=1}^{\infty} |F_D^{n+1}(D^{(n+1)} \Delta^j S^{n+1} x) - F_D^{n+1}(D^{(n+1)} \Delta^j S^n x)|_\infty \\
\leq \sum_{n=1}^{\infty} D^n(\mu_{j+1} - 1) \left( 2CD^{j} \sum_{j \in \Gamma_j} P_1(n)^{\gamma_1} \cdots P_j(n)^{\gamma_j}|\Delta x|_\infty^\gamma \right) < \infty.
\]
Using Lemma 2.2, we have
\[
|F_D^{n+1}(D^{(n+1)} \Delta^j S^n x) - F_D^{n}(D^{jn} \Delta^j S^n x)|_\infty \\
= D^{jn}|F_D^{n+1}(D^{j} \Delta^{j} S^{n} x) - F_D^{n}(D^{j} \Delta^{j} S^{n} x)|_\infty \\
= D^{jn}|F_D^{n+1}(\delta_j \Delta^{j} S^{n} x) - F_D^{n}(\delta_j \Delta^{j} S^{n} x)|_\infty \\
\leq \tilde{C} D^{jn}|\Delta^{j+1} S^n x|_\infty \\
\leq \tilde{C} D^{jn}|D^{(-j-1+\mu_{j+1})n} P_{j+1}(n)|\Delta x|_\infty \\
\leq \tilde{C} D^n(\mu_{j+1} - 1) P_{j+1}(n)|\Delta x|_\infty.
\]
Since \( \mu_{j+1} < 0 \) and \( P_{j+1} \) is a polynomial, it follows that
\[
\sum_{n=1}^{\infty} |F_D^{n+1}(D^{(n+1)} \Delta^j S^n x) - F_D^{n}(D^{jn} \Delta^j S^n x)|_\infty \leq \sum_{n=1}^{\infty} \tilde{C} D^n(\mu_{j+1} - 1) P_{j+1}(n)|\Delta x|_\infty < \infty.
\]
Combining all the above, we obtain
\[
\sum_{n=1}^{\infty} |F_D^{n+1}(D^{(n+1)} \Delta^j S^{n+1} x) - F_D^{n}(D^{jn} \Delta^j S^n x)|_\infty < \infty.
\]
Hence \( F_D^n(D^{jn} \Delta^j S^n x) \) is a Cauchy sequence with respect to \(| \cdot |_\infty\). Therefore \( F_D^n(D^{jn} \Delta^j S^n x) \) converges uniformly as \( n \to \infty \).
Lemma A.3. Let $S$ be a subdivision operator defined on $X_{M, \delta}$ and $\bar{S}$ be a linear subdivision operator such that its derived subdivision operators $S_1, \ldots, S_k$ all exist. Suppose $S$ and $\bar{S}$ have the same dilation factor. If there exists $C_1 > 0$ such that for any $x \in X_{M, \delta}$ and $j = 1, \ldots, k$

$$|\Delta^{j-1}Sx - \Delta^{j-1}\bar{S}x|_\infty \leq C_1 \Omega_j(x).$$

Then for any $n \in \mathbb{N}$, there exists $C_n > 0$ such that for any $x \in X_{M, \delta}$ and $j = 1, \ldots, k$

$$|\Delta^{j-1}S^n x - \Delta^{j-1}\bar{S}^n x|_\infty \leq C_n \Omega_j(x).$$

Proof: Suppose the dilation factor of $S$ and $\bar{S}$ is $D$. Then $\bar{S} \Delta^j \bar{S} = D^j \Delta^j \bar{S}$ for $j = 1, \ldots, k$. Hence

$$\Delta^j \bar{S}^n = D^{-nj} \bar{S}^n \Delta^j, \quad \forall n \in \mathbb{N}, \ j = 1, \ldots, k.$$

We use induction to prove the result. When $n = 1$, the result is true by assumption. Now assume the result is true for $n - 1$. Then for $j = 1, \ldots, k$

$$|\Delta^{j-1}S^n x - \Delta^{j-1}\bar{S}^n x|_\infty \leq |\Delta^{j-1}S^n x - \Delta^{j-1}S S^{n-1}x|_\infty + |\Delta^{j-1}S S^{n-1}x - \Delta^{j-1}\bar{S} S^{n-1}x|_\infty$$

$$\leq C_{n-1} \Omega_j(S^{n-1}x) + D^{1-j} |\bar{S}_{j-1} \Delta^{j-1}S^{n-1}x - \bar{S}_{j-1} \Delta^{j-1}\bar{S}^{n-1}x|_\infty$$

$$\leq C_{n-1} \Omega_j(S^{n-1}x) + D^{1-j} |\bar{S}_{j-1}|_\infty C_{n-1} \Omega_j(x).$$

Since for $j = 1, \ldots, k$,

$$|\Delta^j S^{n-1} x|\infty \leq |\Delta^j \bar{S}^{n-1} x|\infty + |\Delta^j S^{n-1} x - \Delta^j \bar{S}^{n-1} x|\infty$$

$$\leq D^{-(n-1)} |\bar{S}^{n-1} x|\infty \Omega_j(x)$$

$$\leq D^{-(n-1)} |\bar{S}^{n-1} x|\infty |\Delta^j x|\infty + C_{n-1} \Omega_j(x).$$

It follows that

$$\Omega_j(S^{n-1}x) = \sum_{\gamma \in \Gamma_j} |\Delta S^{n-1} x|_{\infty}^{\gamma} \cdots |\Delta^j S^{n-1} x|_{\infty}^{\gamma}$$

$$\leq \sum_{\gamma \in \Gamma_j} \left(D^{-(n-1)} |\bar{S}^{n-1} x|\infty |\Delta x|\infty + C_{n-1} \Omega_j(x)\right)^{\gamma} \left(D^{-(n-1)} |\bar{S}^{n-1} x|\infty |\Delta^j x|\infty + C_{n-1} \Omega_j(x)\right)^{\gamma}.$$

It is easy to verify that for any $\gamma \in \Gamma_j$, there exists $C_{\gamma, n-1} > 0$ such that

$$\left(D^{-(n-1)} |\bar{S}^{n-1} x|\infty |\Delta x|\infty + C_{n-1} \Omega_j(x)\right)^{\gamma} \cdots \left(D^{-(n-1)} |\bar{S}^{n-1} x|\infty |\Delta^j x|\infty + C_{n-1} \Omega_j(x)\right)^{\gamma} \leq C_{\gamma, n-1} \Omega_j(x),$$

where $C_{\gamma, n-1}$ depends on $D$, $\gamma$, $C_{n-1}$ and $|\bar{S}^{n-1} x|\infty$, $\ldots$, $|\bar{S}^j x|\infty$. Therefore

$$\Omega_j(S^{n-1}x) \leq \left(\sum_{\gamma \in \Gamma_j} C_{\gamma, n-1}\right) \Omega_j(x).$$

Hence

$$|\Delta^{j-1}S^n x - \Delta^{j-1}\bar{S}^n x|_\infty \leq C_{n-1} \left(\sum_{\gamma \in \Gamma_j} C_{\gamma, n-1}\right) \Omega_j(x) + D^{1-j} |\bar{S}_{j-1}|_\infty C_{n-1} \Omega_j(x)$$

$$\leq C_n \Omega_j(x),$$

where

$$C_n = C_{n-1} \max_{1 \leq j \leq k} \left(\sum_{\gamma \in \Gamma_j} C_{\gamma, n-1} + D^{1-j} |\bar{S}_{j-1}|_\infty\right).$$
References


Figure 3: Divided difference plots of level $j = 8$ subdivision of Donoho's original log-exp scheme and our modified log-exp scheme on $S^2$-valued data.