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ON DONOHO'S LOG-EXP SUBDIVISION SCHEME: CHOICE OF RETRACTION AND TIME-SYMMETRY*

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Abstract. In recent years a number of different approaches for adapting linear subdivision schemes to manifold-valued data were proposed. In this article, we study the following family:

$$(Sx)_{2i+\sigma} = f_{x_i}\left(\sum_{\ell} a_{2\ell+\sigma}\, g_{x_i}(x_{i-\ell})\right), \quad \sigma = 0, 1, \quad i \in \mathbb{Z};$$

here f is a smooth retraction on M, g is the corresponding local inverse, and $(a_{2\ell+\sigma})$ is the mask of a linear subdivision scheme. This particular way of adapting linear subdivision schemes to manifold, in which the *same base point* x_i is used for both the odd and even rules, is a fundamental building block of the wavelet transform proposed in Ur Rahman et al. [*Multiscale Model. Simul.*, 4 (2005), pp. 1201–1232]. This feature is *not* shared by the other ways proposed in the recent literature. In this article, we expose the rather subtle smoothness equivalence properties of the above S; here "smoothness equivalence property" refers to how much smoothness the nonlinear S inherits from the underlying linear scheme. We first prove that one always gets C^2 equivalence between S and the linear scheme regardless of the choice of f. In contrast, if one wants just one more order of smoothness equivalence, then the choice of f matters. We show that C^3 equivalence is guaranteed by a condition on the third order Taylor expansions of f. This condition is further proved to be genuinely geometric in the sense that it is invariant under change of coordinates. Our second main result shows that the most natural choice $f = \exp$ in a symmetric space setting always satisfies the condition. Consequently, any third order accurate approximation of the exponential map would satisfy the same condition. This provides the ground for replacing the exponential map by a more computationally efficient approximant. The difficulty is that such an approximant must also be chosen such that it is by itself also a retraction of the underlying symmetric space or Lie group. Fortunately, it is a well-studied problem in the area of numerical geometric integration; many computationally efficient approximations to the exponential map are available for different symmetric spaces. Finally, we discuss the effect of time-symmetry on smoothness.

Key words. manifold-valued data, nonlinear subdivision, smoothness, proximity inequality, log-exp scheme, exponential map, tangent bundle, retraction, symmetric space, Lie group, Ado's theorem, numerical geometric integration

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1. Introduction.

Motivated by the burgeoning of different types of manifold-valued data in areas of science and engineering, for example, in diffusion tensor imaging and collaborative motion modeling, a novel framework of nonlinear wavelet transform is introduced in [16] for multiscale representation of such data. Underlying this wavelet transform is the so-called log-exp subdivision scheme

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$$(S_y)_{2i+\sigma} = \exp_{y_i}\left(\sum_\ell \alpha_{2\ell+\sigma} \log_{y_i}(y_{i-\ell})\right), \qquad \sigma = 0, 1. \tag{1.1}$$

The goal of this article is to analyze this nonlinear subdivision scheme more deeply. Before we proceed, let us first discuss some related works that will help motivate this paper. It was once conjectured by Donoho that the above scheme has a so-called smoothness equivalence property: if the linear scheme with mask (a_k) is C^k smooth, then so is S. Numerical experiments indicate, but have not yet proved, that this conjecture is not true. This "counterconjecture" had since stimulated the effort in finding ways to modify the scheme (1.1) in order to satisfy the hoped-for smoothness equivalence property. By now, two such modifications have been found.

In [23], a scheme of the following form was shown to satisfy C^k smoothness equivalence for any k:

$$(Sy)_{2i} = f_{y_i}\left(\sum_\ell a_{2\ell} g_{y_i}(y_{i-\ell})\right), \qquad (Sy)_{2i+1} = f_{y_{i+1/2}}\left(\sum_\ell a_{2\ell+1} g_{y_{i+1/2}}(y_{i-\ell})\right). \tag{1.2}$$

Here, the modified base point $y_{i+1/2}$ has to be chosen based on an auxiliary interpolatory subdivision scheme with approximation order $k+1$, and—particularly relevant to this paper—(f, g) need not be (exp, log) but basically any smooth mappings such that g is a left inverse of f (see below).

Afterward, Grohs [8] found that another approach can be based on modifying the weighted averages in the linear scheme to the corresponding Karcher means [13]. To put things in perspective, it is worth viewing this approach also as a two base point scheme similar to (1.2): $(Sy)_{2i+\sigma} = b_{i+\sigma} = \exp_{b_{i+\sigma}}(\sum_\ell a_{2\ell+\sigma} \log_{b_{i+\sigma}}(y_{i-\ell}))$, $\sigma = 0, 1$, where the two base points b_i, b_{i+1} are the corresponding Karcher means. This rather convoluted description comes from the fact that the Karcher mean of a set of points $\{p_i\}$ on a Riemannian manifold with respect to weights w_i, $\sum_i w_i = 1$, is the limit of a fixed point iteration: $m^{(k+1)} = \exp_{m^{(k)}} \sum_i w_i \log_{m^{(k)}} p_i$, $k = 1, 2, \ldots$.

Compared to (1.2), the Karcher mean approach has the advantage of preserving any symmetry in the underlying linear scheme (see section 5.1) and also that the required smoothness equivalence order k does not need to be known a priori. On the other hand, (1.2) is more computationally efficient than the Karcher mean approach for two reasons: a Karcher mean is usually computed based on the aforementioned fixed point iteration with each iteration having the same complexity as that of the computation of (1.1). In contrast, not only does (1.2) involve just a small number of applications of f and g for each fixed i and σ, but f and g can be chosen to be maps that are computationally cheaper than the exponential and logarithmic maps. Furthermore, the approach in [23] can actually be slightly modified to accommodate symmetry; see footnote 4 in section 5.1.

Such developments may suggest that we should simply abandon the original scheme (1.1). The goal of this paper is quite the opposite: we would like to go back to study the original single base point scheme (1.1).

1.1. Motivation of this paper. In the original construction of manifold-valued data wavelet transform [16], it is quite crucial to use the *same base point* for both the odd and even rules. This is because when the transform coarsens an input data from scale $j+1$ to the next coarser scale j, two data points $f_{j+1,2k}$, $f_{j+1,2k}$, assumed to take values in a symmetric space, are approximated by their midpoint: $\beta_{j,k} := \text{Midpoint}(f_{j+1,2k}, f_{j+1,2k})$; in order to account for the error in this coarsening step, a "wavelet coefficient," based on average interpolation, is calculated, and this wavelet coefficient has to reside in the tangent plane at $\beta_{j,k}$. Unless we want to consider a kind of redundant pyramid representation for manifold-valued data, it is necessary to adhere to the original single base point scheme. On the other hand, we need not restrict ourselves to the exponential map. For many symmetric spaces, there exist retractions that are more computationally efficient compared to the exponential map. Therefore, in this article we focus on studying the following family of single base point schemes:

$$(Sx)_{2i+\sigma} = f_{x_i}\left(\sum_{\ell} a_{2\ell+\sigma} g_{x_i}(x_{i-\ell})\right), \quad \sigma = 0, 1, \quad i \in \mathbb{Z}. \tag{1.3}$$

Here, f is a smooth retraction of the manifold and g is the corresponding local inverse. See section 1.3.

After explaining the more practical motivation, the authors cannot deny that an equally pertinent motivation is purely mathematical: the smoothness properties of the single base point schemes are much more intriguing than we first thought; see the next section.

1.2. Results. Our main results pertaining to (1.3) are Theorems 5 and 8.

Theorem 5, together with the companion numerical evidence, shows that a special condition on the third order Taylor expansion of f ought to be satisfied in order for the single base point scheme (1.3) to enjoy a C^3 equivalence property (see section 1.4 for terminology.) Perhaps surprisingly, no such condition is needed at all for lower order smoothness equivalence.

Although a Taylor expansion is dependent on a choice of local coordinates, the aforementioned condition is further shown to be genuinely geometric in the sense that it is invariant under change of coordinates. Effectively, we discover along the way a somewhat unfamiliar geometric invariant P_f of the retraction map f, with which our condition in Theorem 5 reads $P_f = 0$. See Proposition 6 and the appendix.

Theorem 8 then shows that in the setting of symmetric spaces the exponential map always satisfies the special C^3 condition.

These two results together imply that if one can find a third order accurate approximation to the exponential map, then the approximation will also satisfy the C^3 condition. Such a third order approximation must also be a retraction of the symmetric space by itself. We point to interesting answers to this nontrivial problem discovered in the literature of geometric integration. Results from this area provide us with retractions computationally more efficient than the exponential map without jeopardizing the C^3 equivalence property. See section 4.4.

Finally, we provide numerical evidence that indicates that by imposing a "time-symmetry" to (2.15) one can gain yet one more order of smoothness equivalence. A proof of this fact will be documented elsewhere.

The results in this paper stand in marked contrast to the smoothness equivalence property of the modified base point scheme (1.2) developed in [23]. In the latter, neither the choice of (f, g) nor the time-symmetry plays any role in smoothness: the special choice of the modified base point $y_{i+1/2}$ simply "does all the tricks" for smoothness equivalence. (The computation of $y_{i+1/2}$ still depends on f and g, but again the choice of f and g matters the least. In fact, in [23] the computation of $y_{i+1/2}$ is based on yet another pair of arbitrarily chosen $(\hat{f}, \hat{g})$.) A key message of this paper is that the situation is entirely different for the single base point scheme.

1.3. Setup. Let M be a smooth manifold of dimension n. We let T be a linear subdivision operator defined by

$$(Ty)_{2i+\sigma} = \sum_{\ell} a_{2\ell+\sigma} y_{i-\ell}, \quad \sigma = 0, 1, \quad i \in \mathbb{Z}. \tag{1.4}$$

Our general nonlinear subdivision scheme S for M-valued data will be derived from this linear scheme T as well as a pair of smooth maps

$$f\colon Domain(f) \subseteq TM \to M, \qquad g\colon Domain(g) \subseteq M \times M \to TM$$

that "map back and forth" M and its tangent bundle TM.

Recall that TM has a differentiable structure of dimension $n + n$. In particular, one can sensibly talk about smooth mappings between M and TM. An element in TM is denoted by a tuple of the form (x, v), where $x \in M$ and $v \in T_x M$.

Assume that, for every $x \in M$, there is an open neighborhood U_x of x, and an associated smooth mapping

$$g_x\colon U_x \to T_x M.$$

It is important for us to also assume that the mapping $(x, y) \mapsto g_x(y)$ is *jointly smooth in x and y*. In order for this to make sense, we first impose that

$$Domain(g) := \{(x, y)\colon x \in M, y \in U_x\} \tag{1.5}$$

is open in $M \times M$. In other words, $Domain(g)$ is an open neighborhood of the diagonal of $M \times M$. Now the map g is assumed to be of the form

$$g(x, y) = (x, g_x(y)).$$

In this sense, g maps points in M to vectors in TM. We also assume that $g_x(x) = 0$ $\forall\, x \in M$.

We now specify the properties of f, which is designated to map vectors in TM back to points in M. For each $x \in M$, let E_x be an open set in $V(x)$ that contains $g_x(U_x)$, and we demand also that

$$Domain(f) := \{(x, v)\colon x \in M, v \in E_x\} \tag{1.6}$$

is open in V. Then $f\colon E \to M$ is assumed to be a smooth map that also satisfies

$$f_x(g_x(y)) = y \quad \forall\, x \in M, y \in U_x, \quad f_x := f(x, \cdot). \tag{1.7}$$

Since $g_x(x) = 0$, $f(x, 0) = x\ \forall\, x$.

Remark 1. We are mostly interested in f being a smooth retraction, meaning that f satisfies one more property, namely, the local rigidity condition $Df_x = \mathrm{Id}$. But this condition plays no role in our analysis until we prove the coordinate independence result (Proposition 6) in the appendix.

With all the definitions in place, the general family of subdivision schemes (1.3) is now well defined.

In local coordinates, we can write (1.3) as

$$(Sx)_{2h+\sigma} = f\Big(x_h, \sum_{\ell} a_{2\ell+\sigma} g(x_h, x_{h-\ell})\Big), \quad \sigma = 0, 1, \quad h \in \mathbb{Z}, \tag{1.8}$$

where the (f, g) in (1.3) is now replaced by a pair of "numerical maps":

$$g\colon \{(x, y)\colon x \in K, y \in x + B\} \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \qquad f\colon K \times B' \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \tag{1.9}$$

where K is a bounded open set in $\mathbb{R}^n$, B is a certain open ball in $\mathbb{R}^n$ centered at 0, B' is a certain open ball in $\mathbb{R}^n$ at 0, and $f(x, v) \in x + B \ \forall\ v \in B'$. We also have

$$f(x, g(x, y)) = y \tag{1.10}$$

$\forall\ x \in K$ and $y \in x + B$.

It can be shown that as long as the data $(x_h)_h$ stays inside a small enough subset of K and that the linear scheme is convergent, then all subdivision data $S^j x$ is well defined and stays inside K. See [23].

1.4. C^k Equivalence. Notice that, in our setup, any M-valued subdivision scheme S consists of two ingredients:

1. an underlying linear subdivision scheme T;
2. a specific strategy for defining S based on T.

We emphasize that list items 1 and 2 above are *independent*. For instance, (1.3) can be applied, with everything else fixed, to different linear schemes T. On the other hand, for a fixed linear scheme T, one may adapt it to M-valued data in many different ways; see [18], [22], [21], [23] for the many possibilities. Indeed, even our specific strategy (1.3) has a degree of generality: one may choose many different (f, g) under the same formula.

Therefore, it is more accurate to think of (1.3) as a specific *family* of strategies.

Given any strategy, we say that it has a C^k *equivalence* property if whenever the linear T is C^k smooth, then the corresponding S defined via the strategy is also C^k smooth.

2. Proximity inequality based on block form Taylor expansions.

Our analysis is based on the following "proximity $\Rightarrow$ smoothness" theorem [23, Theorem 2.4].

THEOREM 2. *Assume that the linear scheme T is stable and C^k, $k \geq 1$. If there exists a constant $C > 0$ such that for any dense enough bounded sequence $\mathbf{x}$, we have*

$$\|\Delta^{j-1} S\mathbf{x} - \Delta^{j-1} T\mathbf{x}\|_\infty \leq C\boldsymbol{\Omega}_j(\mathbf{x}), \qquad j = 1, \ldots, k, \tag{2.1}$$

where

$$\boldsymbol{\Omega}_j(\mathbf{x}) := \sum_{\gamma \in \Gamma_j} \prod_{i=1}^{j} \|\Delta^i \mathbf{x}\|_\infty^{\gamma_i}, \qquad \Gamma_j := \Big\{\gamma = (\gamma_1, \ldots, \gamma_j) \Big| \gamma_i \in \mathbb{Z}^+, \sum_{i=1}^{j} i\gamma_i = j + 1\Big\}, \tag{2.2}$$

then S is also C^k.

Remark. We in general call (2.1) a *proximity inequality* or, more precisely, an *order k proximity condition*; a useful feature to bear in mind is the following: For any $k < j$,

$$\|\Delta^k S\mathbf{x} - \Delta^k T\mathbf{x}\|_\infty = O(\Omega_j(\mathbf{x})) \Rightarrow \|\Delta^{j-1} S\mathbf{x} - \Delta^{j-1} T\mathbf{x}\|_\infty = O(\Omega_j(\mathbf{x})).$$

This simply follows from the inequality $\|\Delta^k \mathbf{x}\|_\infty \leq 2^k \|\mathbf{x}\|_\infty$. The converse implication is not true unless S and T are interpolatory.

We analyze the nonlinear scheme (1.8) by comparing it to the linear T:

$$(Tx)_{2h+\sigma} = \sum_\ell a_{2\ell+\sigma} x_{h-\ell}, \quad \sigma = 0, 1, \quad h \in \mathbb{Z}. \tag{2.3}$$

We use the Taylor expansion of f at $(x, 0)$, where x is a certain point in K, in the following "block form":

$$f(x_h, v) = x + \sum_{m=1}^{k} \sum_{\alpha=0}^{m} F_\alpha^{(m)}((x_h - x)^{m-\alpha}, v^\alpha) + O(\max(\|x_h - x\|, \|v\|)^{k+1}), \tag{2.4}$$

where $F_\alpha^{(m)} : (\mathbb{R}^n)^{m-\alpha} \times (\mathbb{R}^n)^\alpha \to \mathbb{R}^n$ is an appropriate multilinear map and we use the shorthand notation

$$F_\alpha^{(m)}(u^{m-\alpha}, v^\alpha) := F_\alpha^{(m)}(\underbrace{u, \ldots, u}_{m-\alpha \text{ times}}, \underbrace{v, \ldots, v}_{\alpha \text{ times}}).$$

Precisely, the above multilinear map is given by[1]

$$\begin{aligned} &F_\alpha^{(m)}(u_1, \ldots, u_{m-\alpha}, v_1, \ldots, v_\alpha) \\ &\quad = \frac{1}{(m-\alpha)!\alpha!} \frac{d}{ds_1}\Big|_{s_1=0} \cdots \frac{d}{ds_{m-\alpha}}\Big|_{s_{m-\alpha}=0} \frac{d}{dt_1}\Big|_{t_1=0} \cdots \frac{d}{dt_\alpha}\Big|_{t_\alpha=0} f\left(x + \sum_{i=1}^{m-\alpha} s_i u_i, \sum_{j=1}^{\alpha} t_j v_j\right). \end{aligned} \tag{2.5}$$

Similarly, we use the following block form Taylor expansion of g at (x, x):

$$g(x_h, y) = \sum_{n=1}^{k_g} \sum_{\beta=0}^{n} G_\beta^{(n)}((x_h - x)^{n-\beta}, (y - x)^\beta) + O(\max(\|x_h - x\|, \|y - x\|)^{k_g+1}), \tag{2.6}$$

$$\begin{aligned} G_\beta^{(n)}\Big(u_1, \ldots, u_{n-\beta}, v_1, \ldots, v_\beta\Big) &= \frac{1}{(n-\beta)!\beta!} \frac{d}{ds_1}\Big|_{s_1=0} \cdots \frac{d}{ds_{n-\beta}}\Big|_{s_{n-\beta}=0} \frac{d}{dt_1}\Big|_{t_1=0} \cdots \frac{d}{dt_\beta}\Big|_{t_\beta=0} \\ &\quad \times g\left(x + \sum_{i=1}^{n-\beta} s_i u_i, x + \sum_{j=1}^{\beta} t_j v_j\right). \end{aligned}$$

If K is bounded, then by the smoothness of f and g one can find constants $C_{\alpha,m}$, $C'_{\beta,n}$ independent of $x \in K$ such that

[1]This notation $F_\alpha^{(m)}$ is easier for the writing of the proofs of our main theorems. In the appendix, we use instead the (more natural) notation $F_{m-\alpha,\alpha} := (m-\alpha)!\alpha! F_\alpha^{(m)}$, which explicitly reminds us that f is differentiated $m - \alpha$ and α times in the first and second arguments, respectively.

$$\|F_\alpha^{(m)}(u_1, \ldots, u_{m-\alpha}, v_1, \ldots, v_\alpha)\| \leq C_{\alpha,m}\|u_1\| \cdots \|u_{m-\alpha}\|\|v_1\| \cdots \|v_\alpha\|, \tag{2.7}$$

$$\|G_\beta^{(n)}(u_1, \ldots, u_{n-\beta}, v_1, \ldots, v_\beta)\| \leq C'_{\beta,n}\|u_1\| \cdots \|u_{n-\beta}\|\|v_1\| \cdots \|v_\beta\|. \tag{2.8}$$

Now we compare S and T based on these Taylor expansions:

$$\begin{aligned}(Sx - Tx)_{2h+\sigma} &= f\left(x_h, \sum_\ell a_{2\ell+\sigma} g(x_h, x_{h-\ell})\right) - \sum_\ell a_{2\ell+\sigma} f(x_h, g(x_h, x_{h-\ell}))\\
&= \sum_{m=2}^k \sum_{\alpha=0}^m F_\alpha^{(m)}\left((x_h - x)^{m-\alpha}, \left(\sum_\ell a_{2\ell+\sigma} g(x_h, x_{h-\ell})\right)^\alpha\right)\\
&\quad - \sum_\ell a_{2\ell+\sigma} \sum_{m=2}^k \sum_{\alpha=0}^m F_\alpha^{(m)}((x_h - x)^{m-\alpha}, g(x_h, x_{h-\ell})^\alpha) + O(\max_h \|x_h - x\|^{k+1})\\
&= \sum_{m=2}^k \sum_{\alpha=2}^m \left\{ F_\alpha^{(m)}\left((x_h - x)^{m-\alpha}, \left(\sum_\ell a_{2\ell+\sigma} g(x_h, x_{h-\ell})\right)\alpha\right)\right.\\
&\quad \left. - \sum_\ell a_{2\ell+\sigma} F_\alpha^{(m)}((x_h - x)^{m-\alpha}, g(x_h, x_{h-\ell})^\alpha)\right\} + O(\max_h \|x_h - x\|^{k+1}).\end{aligned} \tag{2.9}$$

In the above, we used $\sum_\ell a_{2\ell+\sigma} = 1$, the linearity of $F_1^{(m)}$ in the last argument, and $\|g(x', x'')\| \lesssim \max(\|x' - x\|, \|x'' - x\|)$ for $x', x'' \approx x$.

Each entry in the sequence $\Delta^{k-1}(Sx - Tx)$ is determined by a constant, say, L, number of consecutive entries in x. So, there are an i_0 dependent on i and a constant L such that $(\Delta^{k-1}(Sx - Tx))_i$ is determined by $x_{i_0}, x_{i_0+1}, \ldots, x_{i_0+L}$. We choose L big enough to guarantee $L \geq k$.[2] Now for a fixed $i \in \mathbb{Z}$, define $D_\ell \coloneqq \sum_{j=0}^\ell (-1)^{\ell-j}\binom{\ell}{j} x_{i_0+j}$, so we can in turn write each $x_{i_0+\ell}$, $0 \leq \ell \leq L$, in terms of the D_j's via

$$x_{i_0+\ell} = \sum_{j=0}^\ell \binom{\ell}{j} D_j. \tag{2.10}$$

Our goal is to upper-bound the size of $(\Delta^{k-1}(Sx - Tx))_i$ based on the size of D_j for various j. For this purpose, we use the Taylor expansions of f and g at $(x_{i_0}, 0)$. Caution: While the index i, and hence also i_0 (which grows with i), can be arbitrarily large, all the points in the sequences x, Sx, and Tx are assumed to stay within a bounded neighborhood. This in particular means that the point x_{i_0} where we apply the Taylor expansions can potentially be anywhere in this neighborhood.

Define

$$A_j^h \coloneqq \binom{h - i_0}{j}. \tag{2.11}$$

[2]This is automatically true if we assume that T is C^k smooth. This is because the B-spline scheme of degree $k+1$ is the C^k scheme with the smallest support, and for this scheme $L = k$.

When $h \in \{i_0, \ldots, i_0 + L\}$, we can write

$$x_h - x_{i_0} = \sum_{j=1}^{k-1} A_j^h D_j + O(\|\Delta^k x\|_\infty). \tag{2.12}$$

It suffices to analyze separately each term of (2.9) inside $\{\}$, with x replaced by x_{i_0}. For fixed m, α with $2 \le m \ge k$, $2 \le \alpha \le m$, we now rewrite $F_\alpha^{(m)}((x_h - x_{i_0})^{m-\alpha}, (\sum_\ell a_{2\ell+\sigma} g(x_h, x_{h-\ell}))^\alpha)$ and $\sum_\ell a_{2\ell+\sigma} F_\alpha^{(m)}((x_h - x_{i_0})^{m-\alpha}, (g(x_h, x_{h-\ell}))^\alpha)$ using (2.6) with an appropriate k_g. Our goal is to approximate these two terms up to order $O(\|\Delta x\|^{k+1})$. Using $k_g = k$ would work, but in fact we need only $k_g = k - m + 1$. To see this, notice that

$$e \coloneqq x_h - x_{i_0} = O(\|\Delta x\|_\infty)$$

and

$$g(x_h, x_{h-\ell}) = \sum_{n=1}^{k_g} \underbrace{\sum_{\beta=0}^{n} G_\beta^{(n)}((x_h - x_{i_0})^{n-\beta}, (x_{h-\ell} - x_{i_0})^\beta)}_{\coloneqq e_n = O(\|\Delta x\|_\infty^n)} + \underbrace{O(\|\Delta x\|_\infty^{k_g+1})}_{\coloneqq e_{k_g+1}}.$$

Therefore,

$$\begin{aligned} F_\alpha^{(m)}((x_h - x_{i_0})^{m-\alpha}, g(x_h, x_{h-\ell})^\alpha) &= F_\alpha^{(m)}(e^{m-\alpha}, (e_1 + \cdots + e_{k_g+1})^\alpha) \\ &= F_\alpha^{(m)}(e^{m-\alpha}, (e_1 + \cdots + e_{k_g})^\alpha) + \sum_{\text{at least one } i_j \text{ is } k_g+1} F_\alpha^{(m)}(e, \ldots, e, e_{i_1}, \ldots, e_{i_\alpha}). \end{aligned} \tag{2.13}$$

If one of the i_j's is $k_g + 1$, then, since each of the remaining i_j's is at least 1, we have

$$F_\alpha^{(m)}(e, \ldots, e, e_{i_1}, \ldots, e_{i_\alpha}) = O(\|\Delta x\|_\infty^{m-\alpha+k_g+1+\alpha-1}) = O(\|\Delta x\|_\infty^{m+k_g}).$$

So choosing $k_g = k - m + 1$ guarantees that the last term in (2.13) is of the order $O(\|\Delta x\|_\infty^{k+1})$. To conclude,

$$\begin{aligned} &F_\alpha^{(m)}((x_h - x_{i_0})^{m-\alpha}, g(x_h, x_{h-\ell})^\alpha) \\ &\quad = F_\alpha^{(m)}\left((x_h - x_{i_0})^{m-\alpha}, \left[\sum_{n=1}^{k-m+1} \sum_{\beta=0}^{n} G_\beta^{(n)}((x_h - x_{i_0})^{n-\beta}, (x_{h-\ell} - x_{i_0})^\beta)\right]^\alpha\right) \\ &\qquad + O(\|\Delta x\|^{k+1}). \end{aligned} \tag{2.14}$$

By combining (2.14) with (2.12) and the multilinearity of $F_\alpha^{(m)}$, we get

$$\begin{aligned} &F_\alpha^{(m)}((x_h - x_{i_0})^{m-\alpha}, (g(x_h, x_{h-\ell}))^\alpha) \\ &\quad = F_\alpha^{(m)}\left(\left[\sum_{j=1}^{k-1} A_j^h D_j\right]^{m-\alpha}, \left[\sum_{n=1}^{k-m+1} \sum_{\beta=0}^{n} G_\beta^{(n)}\left(\left(\sum_{j=1}^{k-1} A_j^h D_j\right)^{n-\beta}, \left(\sum_{j=1}^{k-1} A_j^{h-\ell} D_j\right)^\beta\right)\right]^\alpha\right) \\ &\qquad + \underbrace{O(\|\Delta^k x\| \cdot \|\Delta x\|) + O(\|\Delta x\|^{k+1})}_{= O(\Omega_k(\mathbf{x}))}. \end{aligned} \tag{2.15}$$

It is then trivial to see that

$$F_\alpha^{(m)}\Big((x_h - x_{i_0})^{m-\alpha}, \Big(\sum_\ell a_{2\ell+\sigma} g(x_h, x_{h-\ell})\Big)^\alpha\Big)$$
$$= F_\alpha^{(m)}\Bigg(\Bigg[\sum_{j=1}^{k-1} A_j^h D_j\Bigg]^{m-\alpha}, \Bigg[\sum_\ell a_{2\ell+\sigma} \sum_{n=1}^{k-m+1}\sum_{\beta=0}^{n} G_\beta^{(n)}\Bigg(\Bigg(\sum_{j=1}^{k-1} A_j^h D_j\Bigg)^{n-\beta}, \Bigg(\sum_{j=1}^{k-1} A_j^{h-\ell} D_j\Bigg)^\beta\Bigg)\Bigg]^\alpha\Bigg)$$
$$+ O(\mathbf{\Omega}_k(\mathbf{x})) \tag{2.16}$$

and

$$\sum_\ell a_{2\ell+\sigma} F_\alpha^{(m)}((x_h - x_{i_0})^{m-\alpha}, (g(x_h, x_{h-\ell}))^\alpha)$$
$$= \sum_\ell a_{2\ell+\sigma} F_\alpha^{(m)}\Bigg(\Bigg[\sum_{j=1}^{k-1} A_j^h D_j\Bigg]^{m-\alpha}, \Bigg[\sum_{n=1}^{k-m+1}\sum_{\beta=0}^{n} G_\beta^{(n)}\Bigg(\Bigg(\sum_{j=1}^{k-1} A_j^h D_j\Bigg)^{n-\beta}, \Bigg(\sum_{j=1}^{k-1} A_j^{h-\ell} D_j\Bigg)^\beta\Bigg)\Bigg]^\alpha\Bigg)$$
$$+ O(\mathbf{\Omega}_k(\mathbf{x})). \tag{2.17}$$

Notice that if we use the multilinearity of $F_\alpha^{(m)}$ and $G_\beta^{(n)}$ at their full power, we can write each of (2.16) and (2.17) as a linear combination of terms of the form

$$F_\alpha^{(m)}(D_J, G_{\beta_1}^{(n_1)}(D_{J_1^1}, D_{J_2^1}), \ldots, G_{\beta_\alpha}^{(n_\alpha)}(D_{J_1^\alpha}, D_{J_2^\alpha})), \tag{2.18}$$

where $J := (j_1, \ldots, j_{m-\alpha})$ with $1 \le j_1, \ldots, j_\alpha \le k-1$ and D_J is a shorthand notation for $(D_{j_1}, \ldots, D_{j_{m-\alpha}})$; similarly $D_{J_1^i}$ and $D_{J_2^i}$ are two ordered lists of the D_j's with lengths $n_i - \beta_i$ and β_i, respectively. In the following, we write

$$A_J^h := \prod_k A_{j_k}^h = \prod_k \binom{h - i_0}{j_k} \tag{2.19}$$

whenever $J = (j_k)_k$ is an ordered list of indices. If we properly arrange the sums involved, we get

$$(2.16) = \sum_J \sum_{n_1,\ldots,n_\alpha} \sum_{\beta_1,\ldots,\beta_\alpha} \sum_{J_1^1,\ldots,J_1^\alpha} \sum_{J_2^1,\ldots,J_2^\alpha} A_J^h \Bigg[\prod_{i=1}^{\alpha} \sum_\ell a_{2\ell+\sigma} A_{J_1^i}^h A_{J_2^i}^{h-\ell}\Bigg] \times (2.18)$$
$$+ O(\mathbf{\Omega}_k(\mathrm{x})),$$
$$(2.17) = \sum_J \sum_{n_1,\ldots,n_\alpha} \sum_{\beta_1,\ldots,\beta_\alpha} \sum_{J_1^1,\ldots,J_1^\alpha} \sum_{J_2^1,\ldots,J_2^\alpha} A_J^h \Bigg[\sum_\ell a_{2\ell+\sigma} \prod_{i=1}^{\alpha} A_{J_1^i}^h A_{J_2^i}^{h-\ell}\Bigg] \times (2.18)$$
$$+ O(\mathbf{\Omega}_k(\mathrm{x})). \tag{2.20}$$

In the sums above, each n_i varies between 1 to $k - m + 1$; each β_i varies between 0 to n_i; and J, J_1^i, and J_2^i vary over all tuples of indices between 1 and $k-1$ with lengths $m - \alpha$, $n_i - \beta_i$, and β_i, respectively. Since we are only concerned with terms that cannot be absorbed into $O(\mathbf{\Omega}_k(\mathbf{x}))$, by (2.7)–(2.8) we need to deal only with the terms in (2.18) with

$$|J| + \sum_{i=1}^{\alpha}(|J_1^i| + |J_2^i|) \leq k, \tag{2.21}$$

where $|J| \coloneqq j_1 + \cdots + j_{m-\alpha}$ and $|J_1^i|$, $|J_2^i|$ are defined similarly. In particular, for any fixed m and α, we need to consider only those $n_1, \ldots, n_\alpha$ that satisfy

$$(m-\alpha) + n_1 + \cdots + n_\alpha \leq k. \tag{2.22}$$

We summarize the calculation above in the following lemma.

LEMMA 3. *For any $k \geq 2$,*

$$\begin{aligned}(S\mathbf{x} - T\mathbf{x})_{2h+\sigma} &= O(\Omega_k(\mathbf{x})) \\ &+ \sum_{m=2}^{k}\sum_{\alpha=2}^{m}\sum_{J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^{\alpha}} c^{h,\sigma}_{J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^{\alpha}} F_\alpha^{(m)}(D_J, G_{\beta_1}^{(n_1)}(D_{J_1^1}, D_{J_2^1}), \ldots, G_{\beta_\alpha}^{(n_\alpha)}(D_{J_1^\alpha}, D_{J_2^\alpha})),\end{aligned} \tag{2.23}$$

where

$$c^{h,\sigma}_{J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^{\alpha}} = \left[A_J^h \prod_{i=1}^{\alpha} A^h_{J_1^i}\right]\left[\prod_{i=1}^{\alpha}\sum_\ell a_{2\ell+\sigma}A^{h-\ell}_{J_2^i} - \sum_\ell a_{2\ell+\sigma}\prod_{i=1}^{\alpha}A^{h-\ell}_{J_2^i}\right]. \tag{2.24}$$

In the third summation in (2.23), *each n_i varies between* 1 *to $k-m+1$; each β_i varies between* 0 *to n_i; J, J_1^i, and J_2^i vary over all tuples of indices between* 1 *and $k-1$ with lengths $m-\alpha$, $n_i-\beta_i$, and β_i, respectively; we can also trim down this summation by considering only the terms that satisfy $n_1 + \cdots + n_\alpha \leq k-m+\alpha$ and $|J| + \sum_{i=1}^{\alpha}(|J_1^i| + |J_2^i|) \leq k$.*

In the rest of this section, we show how to use Lemma 3 to reproduce (and slightly generalize) some old results.

2.1. Arbitrary order smoothness equivalence: (Quasi-)interpolatory case.

THEOREM 4 (a slightly improved [23, Theorem 4.1]). *If the linear scheme reproduces Π_k in the sense that*

$$\sum_\ell a_{2\ell+\sigma}\pi(\ell) = \pi(\sigma/2), \quad \sigma = 0, 1, \quad \forall\, \pi \in \Pi_k, \tag{2.25}$$

then

$$\|S\mathbf{x} - T\mathbf{x}\| = O(\Omega_k(\mathbf{x})). \tag{2.26}$$

Proof. For a fixed h, $p_i(\ell) \coloneqq A^{h-\ell}_{J_2^i}$ is a polynomial of degree $|J_2^i|$ in the argument ℓ. As such, when $\sum_{i=1}^{\alpha}|J_2^i| \leq k$ (which is necessarily true given (2.21)), all p_i's, as well as $\prod_i p_i$, are polynomials of degree not exceeding k, and we have

$$\begin{aligned}&\prod_{i=1}^{\alpha}\sum_\ell a_{2\ell+\sigma}A^{h-\ell}_{J_2^i} - \sum_\ell a_{2\ell+\sigma}\prod_{i=1}^{\alpha}A^{h-\ell}_{J_2^i} \\ &= \prod_{i=1}^{\alpha}\sum_\ell a_{2\ell+\sigma}p_i(\ell) - \sum_\ell a_{2\ell+\sigma}\prod_{i=1}^{\alpha}p_i(\ell) = \prod_i p_i(\sigma/2) - \prod_i p_i(\sigma/2) = 0. \qquad \square\end{aligned} \tag{2.27}$$

The argument in (2.27) is not new and is used in [21], [23]. We have basically only reproved the first part of [23, Theorem 4.1], using a different form of Taylor expansion of f and g. We notice a minor improvement over [23, Theorem 4.1]: In the subdivision literature, the polynomial reproduction condition (2.25) is usually associated only with *interpolatory* subdivision schemes (as is assumed in the first part of [23, Theorem 4.1].) Note that for an interpolatory scheme, $a_{2\ell} = \delta_\ell$, so (2.25) holds automatically for $\sigma = 0$. However, it is not hard to see that a linear subdivision scheme can satisfy (2.25) without being interpolatory. This kind of "quasi-interpolatory" subdivision scheme—one that only interpolates samples of polynomials up to a certain degree—is studied in [6].

2.2. C^1 and C^2 equivalence: Noninterpolatory case. We can also reproduce the low order smoothness (C^1 and C^2) equivalence results in [18], [17] based on what we have gotten so far. Proving C^1 seems trivial now: we simply see from (2.9) that the first sum rule condition $\sum_\ell a_{\sigma+2\ell} = 1$ implies $\|Sx - Tx\|_\infty = O(\|\Delta x\|_\infty^2)$; C^1 equivalence then follows from Theorem 2. The proof of C^2 equivalence, however, is more subtle.

To prove C^2 equivalence, we use Lemma 3 and Theorem 2 with $k = 2$. We look at the relevant coefficients of the form (2.24): since $k = 2$, $m = \alpha = 2$ is the only possibility in the first two summations; consequently length $(J) = m - \alpha = 0$, so J is always the empty list, $n_1 = n_2 = 1$, and β_1, β_2 each can be either 0 or 1. Each of J_1^1, J_2^1, J_1^2, J_2^2 is either the empty list () or the singleton list (1) depending on the values of β_1 and β_2. Below, we drop the subscripts and write $c^{h,\sigma}$ instead of $c^{h,\sigma}_{J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^\alpha}$.

Notice that

$$c^{h,\sigma} = A^h_{J_1^1} A^h_{J_1^2}\left[\left(\sum_\ell a_{2\ell+\sigma} A^{h-\ell}_{J_2^1}\right)\left(\sum_\ell a_{2\ell+\sigma} A^{h-\ell}_{J_2^2}\right) - \sum_\ell a_{2\ell+\sigma} A^{h-\ell}_{J_2^1} A^{h-\ell}_{J_2^2}\right]. \tag{2.28}$$

Since each of J_1^1, J_2^1, J_1^2, J_2^2 is either the empty list () or the singleton list (1), there are four cases to consider. However, note that as long as one of J_2^1 or J_2^2 is the empty list (), then by (2.28) $c^{h,\sigma} = 0$. Therefore, the only nontrivial case is that of

$$\beta_1 = 1, \quad \beta_2 = 1, \quad J_1^1 = (), \quad J_2^1 = (1), \quad J_1^2 = (), \quad J_2^2 = (1).$$

By (2.19), $A^{h-\ell}_{(1)} = -\ell + h - i_0$; therefore

$$\begin{aligned} c^{h,\sigma} &= \left(\sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)\right)^2 - \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)^2 \\ &= \left(\sum_\ell a_{2\ell+\sigma}\ell\right)^2 - \sum_\ell a_{2\ell+\sigma}\ell^2 \neq 0. \end{aligned}$$

The presence of this case suggests that $\|S\mathbf{x} - T\mathbf{x}\| \neq O(\boldsymbol{\Omega}_2(\mathbf{x}))$. However, if we can show that

$$c^{h,1} - c^{h,0} = 0 = c^{h+1,0} - c^{h,1} \tag{2.29}$$

in this particular case, then we have

$$\|\Delta(S\mathbf{x} - T\mathbf{x})\| = O(\boldsymbol{\Omega}_2(\mathbf{x})), \tag{2.30}$$

and C^2 equivalence will follow.

To prove (2.29), we first recall the general sum rule conditions for subdivision schemes: a linear subdivision scheme T with mask (a_ℓ) satisfies sum rules of order $k+1$ if

$$\sum_\ell a_{2\ell}\pi(2\ell) = \sum_\ell a_{2\ell+1}\pi(2\ell+1) \quad \forall\ \pi \in \Pi_k. \tag{2.31}$$

Recall also that it is a necessary condition for T being C^k smooth. Note that (2.31) is equivalent to $\sum_\ell a_{2\ell}\pi(\ell-1/2) = \sum_\ell a_{2\ell+1}\pi(\ell)\ \forall\pi \in \Pi_k$. So when $k=2$, we have

$$\begin{aligned} c^{h,1} - c^{h,0} &= \Big(\sum_\ell a_{2\ell+1}\ell\Big)^2 - \sum_\ell a_{2\ell+1}\ell^2 - \Big(\sum_\ell a_{2\ell}\ell\Big)^2 + \sum_\ell a_{2\ell}\ell^2 \\ &= \Big(\sum_\ell a_{2\ell}(\ell-1/2)\Big)^2 - \sum_\ell a_{2\ell}(\ell-1/2)^2 - \Big(\sum_\ell a_{2\ell}\ell\Big)^2 + \sum_\ell a_{2\ell}\ell^2 \\ &= \Big(\sum_\ell a_{2\ell}\ell\Big)^2 - \sum_\ell a_{2\ell}\ell + 1/4 - \sum_\ell a_{2\ell}\ell^2 + \sum_\ell a_{2\ell}\ell \\ &- 1/4 - \Big(\sum_\ell a_{2\ell}\ell\Big)^2 + \sum_\ell a_{2\ell}\ell^2 \\ &= 0. \end{aligned} \tag{2.32}$$

This proves (2.29); C^2 equivalence then follows from (2.30) and Theorem 2.

3. Conditional C^3 equivalence.

In this section, Δ always refers to the forward difference operator of a sequence. As before, we drop the subscripts in (2.24) and write $c^{h,\sigma}$ instead of $c^{h,\sigma}_{J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^\alpha}$. When we write $\Delta^k(c^{h,\sigma})$, we view $c^{h,\sigma}$ as a sequence whose $(2h+\sigma)$th entry is $c^{h,\sigma}$.

If $c^{h,\sigma}_{J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^\alpha}$ is the zero sequence, we refer to the $(J, (n_i, \beta_i, J_1^i, J_2^i)_{i=1}^\alpha)$ as a *trivial case*. By (2.24), if all but at most one of $J_2^1, \dots, J_2^\alpha$ are empty, i.e., at most one of $\beta_1, \dots, \beta_\alpha$ is zero, then $c^{h,\sigma} = 0$.

As a preparation for the $k=3$ case, consider

$$B_j(\sigma) := \Big(\sum_\ell a_{2\ell+\sigma}\ell\Big)^j - \sum_\ell a_{2\ell+\sigma}\ell^j. \tag{3.1}$$

Of course, $B_1(0) = 0 = B_1(1)$; if the mask satisfies (2.25) with $k=j$, then $B_j(\sigma) = 0$. But in general $B_j(\sigma) \neq 0$.

When T is C^2, the calculation in (2.32) shows

$$B_2(0) = B_2(1) =: B_2. \tag{3.2}$$

If T is C^3 and $(a_{2\ell+\sigma})$ satisfies (2.31) with $k=3$, then a calculation similar to (2.32) shows

$$B_3(0) - B_3(1) = \frac{3}{2}B_2. \tag{3.3}$$

We shall also encounter the term

$$B_{1,2}(\sigma) := \Big(\sum_\ell a_{2\ell+\sigma}\ell\Big)\Big(\sum_\ell a_{2\ell+\sigma}\ell^2\Big) - \Big(\sum_\ell a_{2\ell+\sigma}\ell^3\Big).$$

Using sum rules of order 4,

$$\begin{aligned} B_{1,2}(0) - B_{1,2}(1) &= \Big(\sum_\ell a_{2\ell}\ell\Big)\Big(\sum_\ell a_{2\ell}\ell^2\Big) - \Big(\sum_\ell a_{2\ell}\ell^3\Big) \\ &\quad - \Big(\sum_\ell a_{2\ell}\ell - 1/2\Big)\Big(\sum_\ell a_{2\ell}(\ell - 1/2)^2\Big) + \Big(\sum_\ell a_{2\ell}(\ell-1/2)^3\Big) \\ &= B_2. \end{aligned} \tag{3.4}$$

We now attempt to establish C^3 equivalence using Lemma 3 and Theorem 2 with $k = 3$. When $k = 3$, the only possibilities for (m, α) on the right-hand side of (2.23) are

$$(m, \alpha) = (2, 2), \quad (3, 2), \quad \text{or} \quad (3, 3).$$

3.1. $(m,\alpha) = (2,2)$. We have $J = ()$, $(n_1, n_2) = (1, 1)$, $(1, 2)$, or $(2, 1)$, and $c^{h,\sigma}$ has the same form as (2.28), so $c^{h,\sigma} = 0$ unless $\beta_1, \beta_2 \geq 1$; and we are left with the following three possibilities.

Below, note that $A^h_{(1)} = h - i_0$, $A^{h-\ell}_{(1)} = -\ell + h - i_0$, $A^{h-\ell}_{(1,1)} = (-\ell + h - i_0)^2$, and $A^{h-\ell}_{(2)} = \frac{(-\ell+h-i_0)(-\ell+h-i_0-1)}{2}$.

(I) $(n_1, n_2) = (1, 1)$, $(\beta_1, \beta_2) = (1, 1)$. $J^1_1 = () = J^2_1$.

1. $J^1_2 = (1)$, $J^2_2 = (1)$: This case is the same as that in (2.29); we have $\Delta(c^{h,\sigma}) = 0$.
2. $J^1_2 = (1)$, $J^2_2 = (2)$:

$$\begin{aligned} c^{h,\sigma} &= \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)\sum_\ell a_{2\ell+\sigma}((-\ell+h-i_0)(-\ell+h-i_0-1)/2) \\ &\quad - \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)((-\ell + h - i_0)(-\ell + h - i_0 - 1)/2) \\ &= -(1/2)B_{1,2}(\sigma) + (h - i_0 - 1/2)B_2. \end{aligned} \tag{3.5}$$

3. $J^1_2 = (2)$, $J^2_2 = (1)$: Same as (2).

(II) $(n_1, n_2) = (1, 2)$, $\beta_1 = 1$, $J^1_1 = ()$, $J^1_2 = (1)$.

1. $\beta_2 = 1$, $J^2_1 = (1)$, $J^2_2 = (1)$:

$$\begin{aligned} c^{h,\sigma} &= (h - i_0)\Big[\Big(\sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)\Big)^2 - \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)^2\Big] \\ &= (h - i_0)B_2. \end{aligned} \tag{3.6}$$

2. $\beta_2 = 2$, $J^2_1 = ()$, $J^2_2 = (1, 1)$:

$$\begin{aligned} c^{h,\sigma} &= \Big(\sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)\Big)\Big(\sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)^2\Big) \\ &\quad - \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)^3 \\ &= -B_{1,2}(\sigma) + 2(h - i_0)B_2. \end{aligned} \tag{3.7}$$

(III) $(n_1, n_2) = (2, 1)$, $\beta_2 = 1$, $J^2_1 = ()$, $J^2_2 = (1)$: Same as case (II).

For cases (I)(2) and (I)(3), we have

$$\begin{aligned} &c^{h,0} - 2c^{h,1} + c^{h+1,0} \\ &\quad = -\frac{1}{2}B_{1,2}(0) + \left(h - i_0 - \frac{1}{2}\right)B_2 + B_{1,2}(1) - 2\left(h - i_0 - \frac{1}{2}\right)B_2 \\ &\qquad -\frac{1}{2}B_{1,2}(0) + \left(h - i_0 + \frac{1}{2}\right)B_2 = B_{1,2}(1) - B_{1,2}(0) + B_2 \overset{(3.4)}{=} 0. \end{aligned}$$

Similarly, $c^{h,1} - 2c^{h+1,0} + c^{h+1,1} = B_{1,2}(0) - B_{1,2}(1) - B_2 = 0$. Since the spatial index h is arbitrary in the calculation above, we conclude that

$$\Delta^2(c^{h,\sigma}) = 0.$$

For case (II)(1),

$$\Delta^2(c^{h,\sigma}) = (-1)^\sigma B_2.$$

For case (II)(2),

$$\Delta^2(c^{h,\sigma}) = (-1)^{\sigma+1}2B_2 + (-1)^\sigma 2B_2 = 0.$$

3.2. $(m,\alpha) = (3,2)$. In this case $J = (1)$, $n_1 = n_2 = 1$, and the only nontrivial case is as follows:

(IV) $\beta_1 = \beta_2 = 1,\ J_1^1 = () = J_1^2,\ J_2^1 = (1) = J_2^2$:

$$\begin{aligned} c^{h,\sigma} &= (h - i_0)\left[\left(\sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)\right)^2 - \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)^2\right] \\ &= (h - i_0)B_2 \end{aligned} \tag{3.8}$$

and

$$\Delta^2(c^{h,\sigma}) = (-1)^\sigma B_2.$$

3.3. $(m,\alpha) = (3,3)$. In this case, $J = ()$, $n_1 = n_2 = n_3 = 1$, each β_1, β_2, β_3 is either 0 or 1, and the only nontrivial cases are when at most one of β_1, β_2, β_3 is 0. So there are four nontrivial cases, listed as cases (V1)–(V4) in Table 3.1. In cases (V1)–(V3), we have

$$c^{h,\sigma} = (h - i_0)\left[\left(\sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)\right)^2 - \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)^2\right] = (h - i_0)B_2,$$

so $\Delta^2(c^{h,\sigma}) = (-1)^\sigma B_2$. In case (V3),

$$c^{h,\sigma} = \left(\sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)\right)^3 - \sum_\ell a_{2\ell+\sigma}(-\ell + h - i_0)^3 = -B_3(\sigma) + 3(h - i_0)B_2,$$

so, by (3.3),

Table 3.1
Nontrivial cases when $k = 3$.

Case	(m,α)	J				$\Delta^2(c^{h,\sigma})F_\alpha^{(m)}(D_J, G_{\beta_1}^{(n_1)}(D_{J_1^1}, D_{J_2^1}), \ldots, G_{\beta_\alpha}^{(n_\alpha)}(D_{J_1^\alpha}, D_{J_2^\alpha}))$
	(2,2)		(n_1, n_2)	(β_1, β_2)	$J_1^1,\ J_2^1,\ J_1^2,\ J_2^2$	
(I)			(1,1)			
(I1)		()		(1,1)	(),(1),(),(1)	0
(I2)		()		(1,1)	(),(1),(),(2)	0
(I3)		()		(1,1)	(),(2),(),(1)	0
(II)			(1,2)			
(II1)		()		(1,1)	(),(1),(1),(1)	$(-1)^\sigma B_2 F_2^{(2)}(G_1^{(1)}(D_1), G_1^{(2)}(D_1, D_1))$
(II2)		()		(1,2)	(),(1),(),(1,1)	0
(III)			(2,1)			
(III1)		()		(1,1)	(1),(1),(),(1)	$(-1)^\sigma B_2 F_2^{(2)}(G_1^{(2)}(D_1, D_1), G_1^{(1)}(D_1))$
(III2)		()		(2,1)	(),(1,1),(),(1)	0
(IV)	(3,2)	(1)	(1,1)	(1,1)	(),(1),(),(1)	$(-1)^\sigma B_2 F_2^{(3)}(D_1, G_1^{(1)}(D_1), G_1^{(1)}(D_1))$
	(3,3)		(n_1, n_2, n_3)	$(\beta_1, \beta_2, \beta_3)$	$J_1^1,\ J_2^1,\ J_1^2,\ J_2^2,\ J_1^3,\ J_2^3$	
(V)			(1,1,1)			
(V1)		()		(1,1,0)	(),(1),(),(1), (1),()	$(-1)^\sigma B_2 F_3^{(3)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_1))$
(V2)		()		(1,0,1)	(),(1),(1),(), (),(1)	$(-1)^\sigma B_2 F_3^{(3)}(G_1^{(1)}(D_1), G_0^{(1)}(D_1), G_1^{(1)}(D_1))$
(V3)		()		(0,1,1)	(1),(),(),(1), (),(1)	$(-1)^\sigma B_2 F_3^{(3)}(G_0^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(1)}(D_1))$
(V4)		()		(1,1,1)	(),(1),(),(1), (),(1)	0

$$\Delta^2(c^{h,\sigma}) = (-1)^{\sigma+1}3B_2 + (-1)^\sigma 3B_2 = 0.$$

3.4. Result. In Table 3.1, we summarize all the calculations in sections 3.1–3.3. Since $F_\alpha^{(m)}$ is invariant under permutation of the last α (and the first $m-\alpha$) arguments, cases (II1) and (III1) give the same terms in the $k = 3$ proximity condition, and so do cases (II2) and (III2), and cases (V1)–(V3).

Theorem 5. *S and T satisfy C^3 equivalence if $B_2 = 0$ or*

$$2F_2^{(2)}(G_1^{(1)}(u), G_1^{(2)}(u,u)) + F_2^{(3)}(u, G_1^{(1)}(u)^2) + 3F_3^{(3)}(G_1^{(1)}(u)^2, G_0^{(1)}(u)) = 0 \quad \forall\ u \in \mathbb{R}^n. \tag{3.9}$$

Note that $B_2 = 0$ is a condition pertaining to the linear scheme. A natural way for this condition to occur is that the linear scheme satisfies (2.25) with $k = 2$, but under the latter condition we simply recover only a special case of Theorem 4.

Condition (3.9) can be rewritten solely in terms of the Taylor coefficient of f. See (5.1) in the appendix. Notice also that the left-hand side of (3.9) is an n-vector of degree 3 homogeneous polynomials in the components of u.

Condition (3.9) is based on the Taylor coefficients of f and g written in *some* local coordinates; therefore a basic question is whether the condition is actually coordinate dependent. The smoothness of a manifold-valued function is of course *independent* of the choice of coordinates; therefore one may wish that the same independence holds for our C^3 smoothness condition. In the appendix, we prove that this is indeed the case.

PROPOSITION 6. *Condition* (3.9) *is invariant under change of coordinates.*

Remark 7. Assume that M is regularly embedded in some Euclidean space $\mathbb{R}^N$. Then f and g can be represented by extrinsic coordinates; i.e., f and g now each take two arguments in $\mathbb{R}^N$. If we assume additionally that these f and g can be smoothly extended to appropriate open neighborhoods in $\mathbb{R}^N \times \mathbb{R}^N$ and the multilinear maps in the "extrinsic Taylor expansions" of f and g, taken at $(x, 0)$ and (x, x), respectively, satisfy condition (3.9) for all $x \in \mathbb{R}^N$ on the surface $M \subset \mathbb{R}^N$, then C^3 equivalence between T and S holds. One may think of this use of Theorem 5 as *an extrinsic approach for analyzing an intrinsically defined scheme*; in fact, this is our way of using Theorem 5 to prove our next main result, Theorem 8.

Strictly speaking, when f and g satisfy condition (3.9) in the above extrinsic sense, it does not immediately imply that f and g satisfy (3.9) in the original intrinsic sense. We simply write "f and g satisfy condition (3.9)" when the condition is satisfied in either way.

4. Symmetric spaces. We now show that the condition in Theorem 5 is always satisfied in the case of a symmetric space. We note that there are two definitions of symmetric spaces: one is based on Riemannian metric, and the other is based on Lie group. In the (more classical) Riemannian definition, a symmetric space is a Riemannian manifold whose curvature tensor is invariant under all parallel translations. The Lie-group approach is more general. For compact symmetric spaces, e.g., S^n, $SO(n)$, $G(n, k)$, Stiefel manifolds, etc., the two definitions are actually equivalent. In particular, all these special matrix manifolds possess a Riemannian metric invariant under the corresponding group action. However, even in the compact case the two approaches give rise to very different looking formulas. See section 4.6. An interesting noncompact example emphasized in [20] is the space of d-dimensional affine subspaces in n-space, which is acted upon by the group of rigid motions. Such an affine Grassmannian manifold does not possess an $SE(n)$ invariant metric unless $d = 0$ or $(d, n) = (1, 3)$.

4.1. *GL(n)*. We begin by studying the general linear group $GL(n)$, which is an open submanifold of $\mathbb{R}^{n\times n}$, and we have $g(x, y) = \log(x^{-1}y)$ and $f(x, v) = x \exp(v) = \sum_{r=0}^{\infty} \frac{1}{r!} xv^r$. We now verify condition (3.9). Recall that the multilinear maps $G^{(n)}_\beta$ are supposed to come from a Taylor expansion of g taken at a point of the form (x_0, x_0). When we need to refer to such a multilinear map taken at a general point (x, y), we write $G^{(n)}_\beta|_{(x,y)}$ instead. Recall also that $F^{(m)}_\alpha$ is supposed to come from a Taylor expansion of f taken at a point of the form $(x_0, 0)$.

We have

$$G^{(1)}_0(u) = \frac{d}{ds}\bigg|_{s=0} \log([x_0 + su]^{-1}x_0) = -\frac{d}{ds}\bigg|_{s=0} \log(I + sx_0^{-1}u) = -x_0^{-1}u.$$

Similarly, $G^{(1)}_1(u) = x_0^{-1}u$. In the derivation above, we use the fact that $\log(x^{-1}) = -\log(x)$. By the same fact, $g(x, y) = -g(y, x)$, which implies $G^{(2)}_1|_{(x,y)}(u, v) =$

$\frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0}g(x+su,y+tv) = -\frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0}g(y+tv,x+su) = -G_1^{(2)}|_{(y,x)}(v,u)$. Therefore

$$G_1^{(2)}(u,u) = 0 \quad \forall\ u \in \mathbb{R}^{n\times n}. \tag{4.1}$$

In this case, the first term on the left-hand side of (3.9) always vanishes. We now calculate the following two terms:

$$\begin{aligned} F_3^{(3)}(v_1,v_2,v_3) &= \frac{x_0}{3!}\frac{d}{dt_1}\bigg|_{t_1=0}\frac{d}{dt_2}\bigg|_{t_2=0}\frac{d}{dt_3}\bigg|_{t_3=0}\exp(t_1v_1+t_2v_2+t_3v_3) \\ &= \frac{x_0}{3!3!}(v_1v_2v_3+v_1v_3v_2+v_2v_1v_3+v_2v_3v_1+v_3v_1v_2+v_3v_2v_1), \\ F_2^{(3)}(u,v_1,v_2) &= \frac{1}{2!}\frac{d}{ds}\bigg|_{s=0}\frac{d}{dt_1}\bigg|_{t_1=0}\frac{d}{dt_2}\bigg|_{t_2=0}(x_0+su)\exp(t_1v_1+t_2v_2) \\ &= \frac{1}{2!}u\frac{d}{dt_1}\bigg|_{t_1=0}\frac{d}{dt_2}\bigg|_{t_2=0}\exp(t_1v_1+t_2v_2) = \frac{u}{2!2!}(v_1v_2+v_2v_1). \end{aligned}$$

Therefore, $F_2^{(3)}(u, G_1^{(1)}(u)^2) = \frac{1}{2}ux_0^{-1}ux_0^{-1}u$ and $F_3^{(3)}(G_1^{(1)}(u)^2, G_0^{(1)}(u)) = -\frac{1}{6}ux_0^{-1}ux_0^{-1}u$; so

$$F_2^{(3)}(u, G_1^{(1)}(u)^2) + 3F_3^{(3)}(G_1^{(1)}(u)^2, G_0^{(1)}(u)) = 0$$

$\forall\ u$. Together with (4.1), condition (3.9) is satisfied.

4.2. General symmetric space. As specific as the derivation in the $GL(n)$ case may seem, it is actually the crux of the proof of a more general result.

THEOREM 8. *If $M = G/K$ is a symmetric space and f and g are the corresponding exponential and logarithmic maps, then condition* (3.9) *is satisfied.*

Proof. First, consider the special case where M is a matrix group (i.e., $G \lhd GL(n)$ and $K = \{I\}$). In this case, the log-exp scheme in M is simply the log-exp scheme in $GL(n)$ restricted to a regular submanifold and the corresponding subalgebra of $gl(n)$. Therefore, condition (3.9) follows from the computation in the previous section. Note that this argument is simply Remark 7 in action.

Next, assume that $M = G$ is a general Lie group (i.e., K is the trivial subgroup.) Since subdivision is a local process, and, moreover, the subdivision scheme (1.3) is invariant under the shift operation of G, it suffices to establish condition (3.9) in a neighborhood of the identity element. By Ado's theorem, the Lie algebra of the Lie group G is isomorphic to a subalgebra of $gl(n)$. This is actually equivalent to saying that G is locally isomorphic to a matrix group $H \lhd GL(n)$ for some n. Therefore, in a neighborhood of the identity element, the log-exp scheme in M is really the "same" as a log-exp scheme in a matrix group. Therefore, any smoothness result for a matrix group is automatically transferred to G. This argument was also used in previous related works [18], [19], [7], [21].

Finally, assume $M = G/K$ is a symmetric space, and $b \in M$ is the base point of which the subgroup K leaves invariant. Denote the Lie algebra of G by $\mathfrak{g}$ and that of K by $\mathfrak{k}$. By the definition of symmetric space, there exists a reflection $s\colon \mathfrak{g} \to \mathfrak{g}$ which is also a Lie algebra automorphism, and its $+1$ eigenspace is $\mathfrak{k}$. Since s^2 is the identity,

$\mathfrak{g}$ is the direct sum of the $+1$ and -1 eigenspaces of s. If we denote the -1 eigenspace by $\mathfrak{s}$, then

$$\mathfrak{g} = \mathfrak{s} \bigoplus \mathfrak{k}.$$

(A reminder: $\mathfrak{k}$ is a Lie algebra, but $\mathfrak{s}$ is not.) For any element x in a neighborhood of b in M, there is a unique group element $g \in G$ of the form $g = \exp(\tilde{v})$, $\tilde{v} \in \mathfrak{s}$, such that $g \circ b = x$. This correspondence $x \mapsto \exp(\tilde{v})$ embeds a neighborhood U of $b \in M$ into G. We denote this (local) embedding by

$$I\colon U \subset M \to G. \tag{4.2}$$

This correspondence is also connected to the definition of the Exp and Log maps on the symmetric space M. Given $x \in M$ and $w \in T_xM$, $\mathrm{Exp}_x(w)$ is defined in the following way. Pick an arbitrary $g \in G$ such that $g \circ b = x$; then let $v \in T_bM$ be such that $dg(v) = w$. If we define $\pi\colon G \to M = G/K$ by $\pi(g) = g \circ b$, then the differential of π at the identity of G, restricted to $\mathfrak{s}$, is a linear isomorphism between $\mathfrak{s}$ and T_bM. Therefore $v \in T_bM$ can be identified with the vector $\tilde{v} = [d\pi|_{\mathfrak{s}}]^{-1} v$ in $\mathfrak{s}$. Then $\mathrm{Exp}_x(w) \coloneqq g \exp(\tilde{v}) \circ b$. Note that the choice of g is not unique; however, it can be shown that the definition of Exp is independent of the choice of g [20]. To define $\mathrm{Log}_x(y)$ for a point y in a neighborhood of x, again let $g \in G$ be such that $g \circ b = x$, and let $\tilde{v}$ be the unique vector $\mathfrak{s}$ such that $g \exp(\tilde{v}) \circ b = y$. Then $\mathrm{Log}_x(y) = dg(d\pi(\tilde{v}))$.

Now if we combine the last two paragraphs, we see that if our control points x_i are close enough to b, then

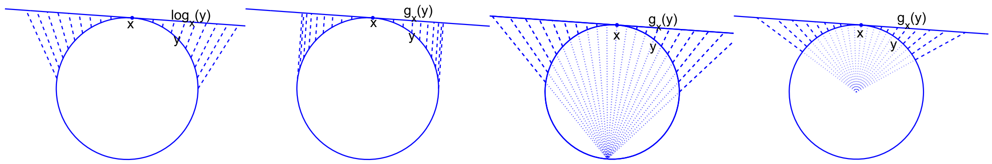
$$I\left(\mathrm{Exp}_{x_i}\left(\sum_\ell a_{2\ell+\sigma} \,\mathrm{Log}_{x_i}(x_{i-\ell})\right)\right) = \exp_{I(x_i)}\left(\sum_\ell a_{2\ell+\sigma} \log_{I(x_i)} I(x_{i-\ell})\right).$$

(The linear combinations on the right-hand side all happen in $\mathfrak{s}$.) This means that the curve produced by the Log-Exp scheme on M is the preimage under I of a curve generated by the log-exp scheme on G. Notice also that I^{-1} has the same action as π, which is smooth (by the very definition of the homogeneous space differentiable structure of $M = G/K$; see, for example, [2]), so the former curve shares the smoothness property of the latter. The assumption that the x_i's are close to b imposes no loss of generality, as we can always find a single group element $g_0 \in G$ that shifts any group of data points that are close to each other to a group of points close to b. To complete this last argument, we should also acknowledge an invariance property of Exp and Log, which guarantees

$$g_0 \circ \mathrm{Exp}_{x_i}\left(\sum_\ell a_{2\ell+\sigma} \,\mathrm{Log}_{x_i}(x_{i-\ell})\right) = \mathrm{Exp}_{g_0 \circ x_i}\left(\sum_\ell a_{2\ell+\sigma} \,\mathrm{Log}_{g_0 \circ x_i}(g_0 \circ x_{i-\ell})\right). \tag{4.3}$$

We have just extended the proof of the theorem from the Lie group case to the case of symmetric space. ∎

4.3. S^n. We begin with the sphere S^n and consider four different retractions on it, depicted by Figure 4.1. In the most elementary $n = 1$ case, denote the natural coordinate of the circle S^1 by ϕ. Then the induced coordinate of the tangent bundle TS^1 is given by $(\phi, \frac{d}{d\phi})$, and we have the following formulas for (g, f):



(a) Exponential map (b) Orthogonal projection (c) Stereographic projection (d) Center projection

FIG. 4.1. *Different retractions on the sphere.*

$$g(\phi,\theta) = \begin{cases} \theta-\phi, \\ \sin(\theta-\phi), \\ 2\,\tan\left(\frac{\theta-\phi}{2}\right), \\ \tan(\theta-\phi), \end{cases}$$

$$f(\phi,v) = \begin{cases} \phi+v, & \text{exponential map,} \\ \phi+\text{arc}\,\sin(v), & \text{orthogonal projection,} \\ \phi+2\,\text{arc}\,\tan\left(\frac{v}{2}\right), & \text{stereographic projection,} \\ \phi+\text{arc}\,\tan(v), & \text{center projection.} \end{cases} \tag{4.4}$$

These retractions agree to the first order. Moreover, in all four cases

$$\left.\frac{\partial g}{\partial \phi}\right|_{(\phi_0,\phi_0)} = -1, \quad \left.\frac{\partial g}{\partial \theta}\right|_{(\phi_0,\phi_0)} = 1, \quad \left.\frac{\partial^2 g}{\partial \phi \partial \theta}\right|_{(\phi_0,\phi_0)} = 0, \quad \left.\frac{\partial^2 f}{\partial v^2}\right|_{(\phi_0,0)} = 0, \quad \left.\frac{\partial^3 f}{\partial \phi \partial^2 v}\right|_{(\phi_0,0)} = 0.$$

However,

$$\left.\frac{\partial^3 f}{\partial v^3}\right|_{(\phi_0,0)} = \begin{cases} 0, & \text{exponential map,} \\ 1, & \text{orthogonal projection,} \\ -\frac{1}{2}, & \text{stereographic projection,} \\ -2, & \text{center projection.} \end{cases}$$

Therefore, condition (3.9) is satisfied for the exponential map but not for the other retractions. Of course the former is not surprising, as we know a priori that with the exponential retraction on S^1 the "nonlinear" scheme S is simply the linear scheme T applied to the angular variable ϕ and hence satisfies C^k equivalence for arbitrary k in a trivial way. (This is a direct consequence of the Abelian Lie-group structure of S^1, a structure not shared by S^n for $n > 1$.) Moreover, numerical results suggest that the other three retractions all fail to produce C^3 limit functions.

In contrast, Theorem 8 says that the exponential map on $S^n = SO(n+1)/SO(n)$ satisfies condition (3.9) for any dimension n. When n is larger than 1, C^k $(k > 2)$ equivalence is not guaranteed. Numerical results suggest that we get C^3 but no higher order smoothness equivalence in general.

Instead of locally embedding S^n into $SO(n+1)$ and reducing the verification of condition (3.9) to the $GL(n)$ calculation in section 4.1, one may use the standard embedding of S^n into $\mathbb{R}^{n+1}$, and the Log and Exp maps are given by[3]

[3]See section 4.6 for a discussion of Lie-group versus Riemannian definitions of the exponential map.

$$\mathrm{Exp}_x(v) = \cos(\|v\|)x + \sin(\|v\|)\frac{v}{\|v\|}, \qquad \mathrm{Log}_x(y) = \arccos(\langle x, y\rangle)\frac{y - \langle x, y\rangle x}{\|y - \langle x, y\rangle x\|}. \tag{4.5}$$

To use Theorem 5 and its companion Remark 7, one has to first extend the definitions of these two maps to suitable open neighborhoods of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, e.g., modify the expression arc $\cos(\langle x, y\rangle)$ to arc $\cos(\langle x, y\rangle / \|x\|\|y\|)$ in order to allow for points x, y not on the sphere. This approach actually works but leads to a relatively more tedious calculation.

4.4. *SO*(*n*), *Sp*(*n*), *SL*(*n*). We first compare the standard exponential map on $SO(n)$ with the following retraction derived from the Cayley transform:

$$f_{\mathrm{Cay}}(x, v) := x\left(I + \frac{1}{2}v\right)\left(I - \frac{1}{2}v\right)^{-1}, \qquad g_{\mathrm{Cay}}(x, y) := -2(I - x^{-1}y)(I + x^{-1}y)^{-1}.$$

To show that $f_{\mathrm{Cay}}(x, \cdot)$ and $g_{\mathrm{Cay}}(x, \cdot)$ are inverse of each other, use the fact that $(I - A)$ and $(I + A)^{-1}$ commute. Note that $g_{\mathrm{Cay}}(x, y) = 2(I + x^{-1}y)^{-1} - 2(I + y^{-1}x)^{-1}$, so $g_{\mathrm{Cay}}(x, y) = -g_{\mathrm{Cay}}(y, x)$, and therefore (4.1) is satisfied also by g_{Cay}. However, one can check, using the same kind of calculation in section 4.1, that $(f_{\mathrm{Cay}}, g_{\mathrm{Cay}})$ does not satisfy condition (3.9). Experiments also suggest that it does not generate C^3 curves.

We further note that, for a small enough v, $f_{\mathrm{Cay}}(x, v) = x(I + v + \frac{1}{2}v^2 + \frac{1}{4}v^3 + \cdots)$, so it agrees with $x\exp(v) = x(I + v + \frac{1}{2}v^2 + \frac{1}{6}v^3 + \cdots)$ up to the second degree term. On the other hand, the Cayley transform $A \mapsto (I + A)(I - A)^{-1}$ maps $so(n)$ exactly, not approximately, to $SO(n)$.

Higher order approximations of the exponential map, while preserving the $so(n) \to SO(n)$ property, exist. For instance, one can get arbitrarily high order approximation based on the Padé rational approximants $R_{m,m}(z)$ of e^z. For example,

$$\underbrace{\frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}}_{R_{1,1}(z)} + O(z^3) = e^z = \underbrace{\frac{1 + \frac{1}{2}z + \frac{1}{12}z^2}{1 - \frac{1}{2}z + \frac{1}{12}z^2}}_{R_{2,2}(z)} + O(z^5),$$

etc. These so-called diagonal Padé approximants can be used to construct retractions of $SO(n)$. $R_{1,1}$ simply recovers the Cayley transform; $R_{2,2}$ can be used to construct a retraction that satisfies our C^3 condition. For this purpose, consider also the $(2, 2)$-Padé approximant of $\log(z)$: $\log(z) = 3(-1 + z^2)/(1 + 4z + z^2) + O((1 - z)^5)$, which is shown in [4] to map $SO(n)$ to $so(n)$; then define

$$f_{\mathrm{Padé}(2,2)}(x, v) := x\left(I + \frac{1}{2}v + \frac{1}{12}v^2\right)\left(I - \frac{1}{2}v + \frac{1}{12}v^2\right)^{-1},$$
$$g_{\mathrm{Padé}(2,2)}(x, y) := 3(-I + (x^{-1}y)^2)(I + 4x^{-1}y + (x^{-1}y)^2).$$

Since $(f_{\mathrm{Padé}(2,2)}, g_{\mathrm{Padé}(2.2)})$ approximates (exp, log) up to a high enough order, it automatically satisfies condition (3.9). However, $f(x, \cdot)$ and $g(x, \cdot)$ are not exact inverses of each other. But this does not really affect the C^3 result, as we have $f(x, g(x, y)) = y + O(\|x - y\|^5)$; by inspecting (2.9), we see immediately that $f(x, \cdot)$ need only be an accurate enough approximate left inverse of $g(x, \cdot)$ in order for the argument (and also the algorithm itself!) to work.

Computation confirms that the $(2,2)$-Padé approximations indeed preserve C^3 equivalence, and they are faster than an implementation based on the standard matrix exponential and logarithm functions `expm` and `logm` in MATLAB. We observed a speedup of 2–3 times in the case of $SO(3)$, and 4–5 times in the case of $SO(5)$. As expected, the factor of speedup increases with the dimension n.

The same diagonal Padé approximations can be applied to the symplectic group: if A is Hamiltonian, i.e., $A^TJ + JA = 0$, $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, then $T = R_{m,m}(A)$ is symplectic, i.e., $T^TJT = J$.

However, diagonal Padé approximations do not work for all Lie groups. In fact, a result by Kang and Shang [12, Lemma 1] shows that for the group of volume-preserving linear maps ($SL(n)$) no analytic approximations of the exponential map can exist: the only analytic function $\phi(z)$ that is consistent with the exponential function up to the first degree, i.e., $\phi(z) = 1 + z + O(z^2)$, and satisfies $\phi(sl(n)) \subset SL(n)$ is the exponential function itself. However, nonanalytic approximations are possible; an approach based on generalized polar decomposition [14], [15], [3] is found to be more powerful than Padé approximation.

Approximating the exponential map while preserving an underlying Lie-group structure is a topic of primary interest in numerical geometric integration [9], [10], [3], [11] as well as in numerical optimization on manifolds [1].

4.5. $G(n,k)$. We now consider the Grassmann manifold $G(n,k)$ of (unoriented) k-dimensional subspaces in $\mathbb{R}^n$. Instead of the Lie-group formulation of symmetric spaces, which identifies $G(n,k)$ with $O(n)/(O(k)\times O(n-k))$, we use a Riemannian setup, which identifies $G(n,k)$ with $\mathbb{R}_*^{n\times k}/GL(k)$, where $\mathbb{R}_*^{n\times k}$ is the manifold of full rank $n\times k$ real matrices (also known as the noncompact Stiefel manifold) and the "division by $GL(k)$" means we identify two matrices $A, B \in \mathbb{R}_*^{n\times k}$ if $\operatorname{span}(A) = \operatorname{span}(B)$, i.e., $A = BG$ for some $G \in GL(k)$.

Under the latter identification, $G(n,k)$ can be turned into a Riemannian symmetric space based on first defining the Riemannian metric $g_A(Z_1, Z_2) := \operatorname{trace}((A^TA)^{-1}Z_1^TZ_2)$ on $\mathbb{R}_*^{n\times k}$ and then descend it to $G(n,k)$; see [1, Proposition 3.6.1]. Under this framework, the equations for geodesics can be given in a very explicit form. For a given $A \in \mathbb{R}_*^{n\times k}$, if we identify the tangent space of $G(n,k)$ at $\operatorname{span}(A)$ by $T_{\operatorname{span}(A)}G(n,k) = \{Z \in \mathbb{R}^{n\times k} : A^TZ = 0\}$, then the Exp and Log maps are given by

$$\operatorname{Exp}_{\operatorname{span}(A)}(Z) = \operatorname{span}(A(A^TA)^{-\frac{1}{2}}V\cos(\Sigma) + U\sin(\Sigma)), \tag{4.6}$$

where $U\Sigma V^T$ is the thin SVD of Z, and

$$\operatorname{Log}_{\operatorname{span}(A)}(\operatorname{span}(B)) = U\tan^{-1}(\Sigma)\,V^T, \tag{4.7}$$

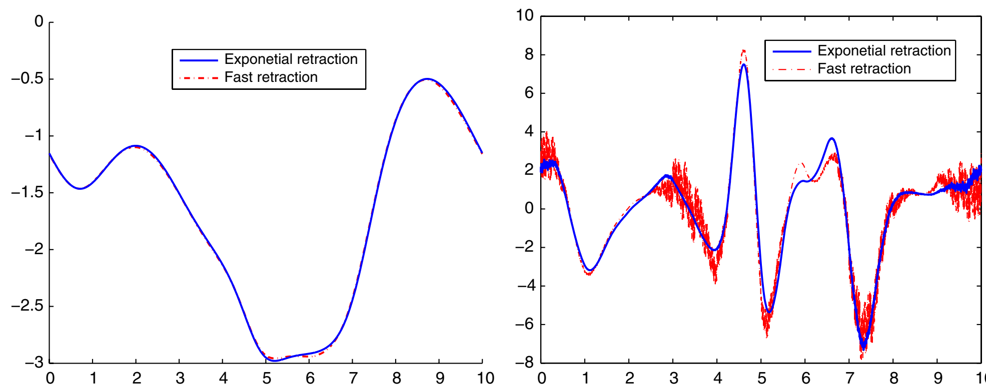
where $U\Sigma V^T$ is the thin SVD of $(I_n - A(A^TA)^{-1}A^T)B(A^TB)^{-1}(A^TA)^{1/2}$.

For a more computationally efficient retraction, consider

$$f(\operatorname{span}(A), Z) = \operatorname{span}(A+Z), \quad g(\operatorname{span}(A), \operatorname{span}(B)) = B(A^TB)^{-1}A^TA - A.$$

We shall simply refer to this as the "fast retraction." However, once again computational experiments suggest that a subdivision scheme based on the fast retraction does not satisfy C^3 equivalence; see below.

We quite arbitrarily choose a 10-periodic sequence in $G(4,2)$ and subdivide it based on the degree 5 B-spline and the two retractions above. In Figure 4.2, we plot the $(1,1)$-



(a) limit curves, plotting the $(1,1)$-entry of ortho-projector representations

(b) 3rd order divided difference

FIG. 4.2. *Comparison of $G(4,2)$ subdivision schemes based on two retractions. The fast retraction appears to fail C^3 equivalence, while the exponential retraction is proved to satisfy C^3 equivalence.*

entry of the orthoprojector representations and the third order divided differences. (Note that the orthogonal projector representation of a subspace, i.e., $\text{span}(A) \mapsto A(A^TA)^{-1}A^T$, is an embedding of $G(n,k)$ into the space of $n \times n$ symmetric matrices.)

4.6. A note on Lie-group versus Riemannian symmetric spaces. After seeing a number of examples and also the proof of Theorem 8, it may be interesting to note how the Lie-group and Riemannian definitions of symmetric spaces lead to seemingly unrelated formulas for the Exp and Log maps. If one uses the Lie-group definition, one identifies the tangent spaces of $G(n,k)$ with the following subspace of $so(n)$:

$$\left\{ \begin{bmatrix} 0_k & P \\ -P^T & 0_{n-k} \end{bmatrix} : P \in \mathbb{R}^{k\times(n-k)} \right\}.$$

This is the space $\mathfrak{s}$ in the proof of Theorem 8. Computing the Log map amounts to finding, for each $p \in G(n,k)$ in a neighborhood of $b = \text{span}(e_1, \ldots, e_k)$, the unique $SO(n)$ element of the form $r = \exp(\left[\begin{smallmatrix} 0 & P \\ -P^T & 0 \end{smallmatrix}\right])$ such that $r \circ b = p$. This one-to-one correspondence $P \leftrightarrow p$ defines the Exp and Log maps. While its wildly different looking Riemannian counterpart (4.6)–(4.7) is what we use in our computational experiments, the Lie-group definition is the key to our proof of Theorem 8.

A similar comment applies to the n-sphere. If one thinks of S^n as $SO(n+1)/SO(n)$, as we did in the proof, then the symmetric space Exp map is related to the matrix exp map in the following way: For any point $p \in S^n$ in a neighborhood of the "north pole" $b = [0, \ldots, 0, 1]^T$, to find $\text{Log}_b(p)$, look for the unique rotation of the form $r = \exp(\left[\begin{smallmatrix} 0_n & v \\ -v^T & 0 \end{smallmatrix}\right])$ that maps the north pole to the point p, i.e., $r \circ b = p$. Then this one-to-one correspondence

$$S^n \ni p \leftrightarrow v \in \mathbb{R}^n$$

defines the Exp and Log maps for S^n. Again, compare this with its Riemannian counterpart (4.5).

Although the two definitions are ultimately equivalent mathematically for compact spaces, they are likely to have different numerical behaviors.

5. Conclusion and open questions. An obvious open question is whether the sufficient condition (3.9) in Theorem 5 is indeed necessary in a certain sense. Several examples and experiments appear to support such a conjecture. Since our condition is a consequence of the general proximity condition for nonlinear subdivision schemes, a similar open question can be posed in the latter, more general, setup.

The reason why this paper is devoted mainly to C^3 analysis is that third order differentiability is the first order in which the choice of retraction would make a difference. Also our experiments suggest that the single base point scheme in general does not satisfy any higher smoothness equivalence order, except in the presence of time-symmetry; see section 5.1. Our second main result, Theorem 8, illustrates that in a symmetric space setting the exponential map always satisfies the special condition in Theorem 5. A natural open question here is whether there exists a fundamentally different retraction that satisfies the same condition, or, more generally, what are *all* the solutions to the differential relation (3.9)?

Proposition 6 suggests that there should be a more geometric description of condition (3.9) in Theorem 5. In particular, there should be a more intrinsic/coordinate-free reformulation of the homogeneous polynomial $P(u)$ in (5.1)/(3.9); it almost sounds like P is "measuring" something geometric about the retraction f. A different thought, due to Duchamp, is that since the proof of Theorem 5 is based on the structure

$$\text{condition}\,(3.9) \Rightarrow \text{order 3 proximity condition} \Rightarrow C^3\,\text{equivalence},$$

perhaps first asking whether the proximity condition itself is coordinate independent will help us to understand what is truly going on. This is a question of independent interest.

Remark 7 offers yet another open question: what are the relations between the condition (3.9) being satisfied intrinsically and extrinsically? Does one imply the other?

5.1. Time-symmetry and C^4. We have exploited in the proof of Theorem 8 the symmetry property (4.3) of the Log-Exp scheme, which is also a desirable property for practical purposes. Indeed all the alternative retractions proposed in section 4 preserve this symmetry. While this symmetry pertains to an invariance under transformation in the *range* space; we now discuss the symmetry under a "time" reversal $t \mapsto -t$ in the *domain*.

For this purpose, we have to first assume that the linear scheme has a time-symmetry. If $a_{-i} = a_i$ in (1.4), such as the odd degree B-splines or Deslauriers–Dubuc schemes, then the subdivision process satisfies $T^\infty(y^\sharp)(t) = (T^\infty y)(-t)$, where $y^\sharp$ is the time reversed sequence of y, i.e., $y^\sharp_k = y_{-k}$. If $a_{1-i} = a_i$, such as the even degree B-splines or the average-interpolating subdivision schemes, then the subdivision process satisfies $T^\infty(y^\sharp)(t) = (T^\infty y)(1-t)$. Following the terminology in computer aided geometric design, we refer to the former and latter symmetries as "primal" and "dual," respectively.

If T has a dual symmetry, then so does S, which can now be written as

$$(Sy)_{2i} = \mathrm{Exp}_{y_i} \sum_\ell a_{2\ell} \,\mathrm{Log}_{y_i}\, y_{i-\ell}, \qquad (Sy)_{2i+1} = \mathrm{Exp}_{y_i} \sum_\ell a_{-2\ell} \,\mathrm{Log}_{y_i}\, y_{i-\ell}.$$

The same is *not* true for primal symmetry. One way to construct a manifold analogue of T which also inherits its primal symmetry is to use a different base point $y_{i+1/2}$ for the

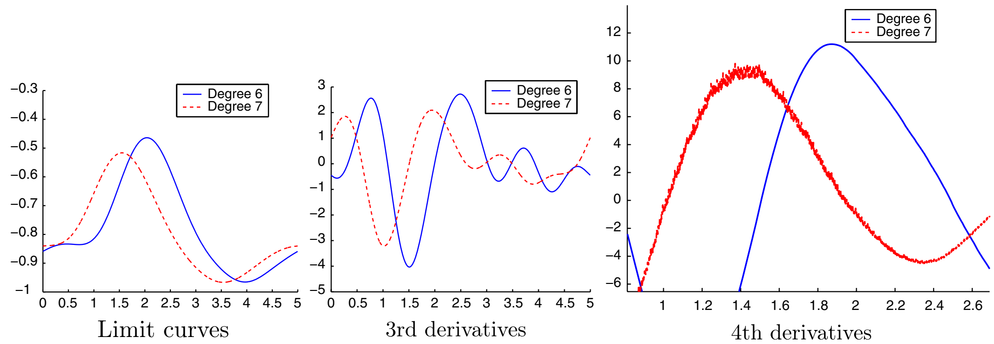


FIG. 5.1. *Dual (time-)symmetry gives the* Log-Exp *scheme an extra order of smoothness equivalence.*

odd $\sigma = 1$ rule, and this base point has to be chosen symmetrically; e.g., in a symmetric space it can be chosen to be the midpoint of the y_i and y_{i+1},[4] i.e.,

$$(Sy)_{2i} = \mathrm{Exp}_{y_i} \sum_{\ell>0} a_{2\ell}(\mathrm{Log}_{y_i} y_{i-\ell} + \mathrm{Log}_{y_i} y_{i+\ell}),$$

$$y_{i+1/2} = \mathrm{Exp}_{y_i} \frac{1}{2} \mathrm{Log}_{y_i} y_{i+1}, \quad (Sy)_{2i+1} = \mathrm{Exp}_{y_{i+1/2}} \sum_{\ell\geq 0} a_{2\ell+1}(\mathrm{Log}_{y_{i+1/2}} y_{i-\ell} + \mathrm{Log}_{y_{i+1/2}} y_{i+\ell}).$$

However, this will defeat our original purpose of studying the single base point scheme (1.3). Therefore, we focus on dual symmetry for now.

Serendipitously, symmetry appears to offer an extra order of smoothness equivalence. While Theorem 8 and our numerical studies suggest that in general Donoho's Log-Exp satisfies only C^3 equivalence, in the presence of dual symmetry the same scheme occurs to satisfy C^4 equivalence. In an unpublished work by Xie, this experimental observation (see below) was proved in the case of $GL(n)$ based on a computer-automated check for proximity conditions. It should be possible to combine this calculation with the argument in the proof of Theorem 8 to extend Xie's proof to a general symmetric space.

As a simple illustration, we use the Log-Exp scheme on $SO(3)$ with the linear scheme being the degree 6 B-spline scheme, $a_k = \binom{7}{k+3}/2^6$, $k = -3, \ldots, 4$, and degree 7 B-spline, $b_k = \binom{8}{k+4}/2^7$, $k = -4, \ldots, 4$. When used with (1.3), the dual symmetry of (a_k) is inherited to S, but S fails to inherit the primal symmetry of (b_k). We apply the two nonlinear schemes to the same set of $SO(3)$ initial data; the resulting curves and their third and fourth order divided difference plots are shown in Figure 5.1. As our main result shows, both nonlinear schemes are C^3 smooth; however, the smoother linear scheme appears to be less smooth when used in the nonlinear setting.

A similar experiment was carried out in the case of the Grassmann manifold and the exponential and log maps (4.6)–(4.7); the same phenomenon was observed.

5.2. Beyond symmetric spaces. While our proof of Theorem 8 is heavily dependent on a symmetric space structure, we suspect that the result itself may actually have

[4]This $y_{i+1/2}$ is not to be confused with the $y_{i+1/2}$ in (1.2). They are designed for different purposes: primal symmetry for the former, and arbitrary order smoothness equivalence for the latter. However, a single $y_{i+1/2}$ that serves both purposes can be constructed.

little to do with symmetric spaces. An open question is whether the same result holds on a general Riemannian manifold.

Experiments on surfaces of revolution shed some light on this question. The computation of exp and log is rather simple for a surface of revolution $\mathbf{x}(u, v) = (\varphi(v)\cos(u), \varphi(v)\sin(u), \psi(v))$ because its Christoffel symbols Γ_{ij}^k can be easily expressed in terms of the univariate functions φ and ψ [5, section 4.3]. The exp and log maps can then be accurately computed by using an initial value and a boundary value ODE solver (e.g., `ode45` and `bvp4c` in MATLAB). We have experimented with such subdivision curves on the surfaces of revolutions. Numerical experiments give exactly the same kind of plots as Figure 4.2, suggesting a subtle role played by the geodesic reproducing property of the log-exp scheme in its smoothness properties and, of course, a possible generalization of Theorem 8 beyond symmetric spaces.

Appendix. Coordinate independence of condition (3.9). In this appendix, instead of writing $F_\alpha^{(m)}$ as defined in (2.5), we use the notation

$$F_{m-\alpha,\alpha} := (m-\alpha)!\alpha! F_\alpha^{(m)}.$$

To prove the coordinate independence of condition (3.9), we first rewrite the condition solely in terms of f. By the identity $f(x, g(x, y)) = y$, we have $F_{0,1} G_1^{(1)} = \mathrm{Id}$, $F_{1,0} + F_{0,1} G_0^{(1)} = 0$, and

$$F_{0,1} G_1^{(2)}(\cdot, \star) + F_{0,2}(G_0^{(1)} \cdot, G_1^{(1)} \star) + F_{1,1}(\cdot, G_1^{(1)} \star) = 0.$$

Since $f(x, 0) = x$, we have

$$F_{1,0} = \mathrm{Id}.$$

Being a retraction, f satisfies the local rigidity condition, so

$$F_{0,1} = \mathrm{Id},$$

which also implies

$$F_{1,1} = 0.$$

Therefore, we can rewrite condition (3.9) as

$$P(u) := F_{0,2}(u, F_{0,2} u^2) + \frac{1}{2} F_{1,2} u^3 - \frac{1}{2} F_{0,3} u^3 \equiv 0. \tag{5.1}$$

Assume that we express the retraction f in two charts (U, ϕ) and (V, ψ). Denote the numerical realizations of f under the two charts by F and $\tilde{F}$ and their Taylor coefficients by $F_{m-\alpha,\alpha}$ and $\tilde{F}_{m-\alpha,\alpha}$. Denote by $\chi : \phi(U \cap V) \to \psi(U \cap V)$ the corresponding change of coordinates map. Then the transformation rule between F and $\tilde{F}$ is

$$F(x, v) = \chi^{-1} \circ \tilde{F}(\chi(x), [\chi'(x)](v)). \tag{5.2}$$

Assume that $\mathbf{x} \in U \cap V$, $x = \phi(\mathbf{x})$, and $\tilde{x} = \psi(\mathbf{x}) = \chi(x)$. We now compare the two degree 3 homogeneous polynomials P and $\tilde{P}$ from (5.1) based on the Taylor polynomials of F at x and $\tilde{F}$ at $\tilde{x}$. The goal is to prove that P vanishes if and only if $\tilde{P}$ vanishes.

Armed with the transformation rule (5.2), we obtain by the chain rule that

$$
\begin{aligned}
F_{0,2} &= (\chi^{-1})''\chi'^2 + (\chi^{-1})'\tilde{F}_{0,2}\chi'^2,\\
F_{0,3} &= (\chi^{-1})'''\chi'^3 + 3(\chi^{-1})''(\tilde{F}_{0,2}(\chi'^2), \chi') + (\chi^{-1})'\tilde{F}_{0,3}\chi'^3,\\
F_{1,2} &= (\chi^{-1})'''\chi'^3 + 2(\chi^{-1})''(\chi', \chi'')\\
&\quad + (\chi^{-1})''(\chi', \tilde{F}_{0,2}(\chi'^2)) + (\chi^{-1})'\tilde{F}_{1,2}(\chi'^3) + 2(\chi^{-1})'\tilde{F}_{0,2}(\chi', \chi''). \qquad (5.3)
\end{aligned}
$$

Above, we use $\tilde{F}_{1,0} = \tilde{F}_{0,1} = \mathrm{Id}$ and $\tilde{F}_{1,1} = 0$ to simplify some of the terms. The derivatives $F_{m-\alpha,\alpha}$, $\tilde{F}_{m-\alpha,\alpha}$, $\chi^{(m)}$, and $(\chi^{-1})^{(m)}$ are evaluated at $(x, 0)$, $(\tilde{x}, 0)$, x, and $\tilde{x}$, respectively; they are also understood to be m-linear maps from $(\mathbb{R}^n)^m$ to $\mathbb{R}^n$. As before, we use the shorthand notation u^m to stand for $(u, \ldots, u)$ whenever we want to save space.[5] So, for example, the expression for $F_{0,3}$ above means

$$
\begin{aligned}
F_{0,3}(u, v, w) &= (\chi^{-1})'''(\chi'u, \chi'v, \chi'w) + 3(\chi^{-1})''(\tilde{F}_{0,2}(\chi'u, \chi'v), \chi'w)\\
&\quad + (\chi^{-1})'\tilde{F}_{0,3}(\chi'u, \chi'v, \chi'w),
\end{aligned}
$$

although we shall ultimately be only interested in the homogeneous polynomial $F_{0,3}(u, u, u)$ in u. Similar comments apply to the other expressions.

Now let us witness the cancellations that lead to the proof of Proposition 6:

$$
\begin{aligned}
P(\cdot) =& F_{0,2}(\cdot, F_{0,2}(\cdot,\cdot)) + \frac{1}{2}F_{1,2}(\cdot,\cdot,\cdot) - \frac{1}{2}F_{0,3}(\cdot,\cdot,\cdot)\\
=& (\chi^{-1})''(\chi', \chi'[(\chi^{-1})''\chi'^2 + (\chi^{-1})'\tilde{F}_{0,2}\chi'^2])\\
& + (\chi^{-1})'\tilde{F}_{0,2}(\chi', \chi'[(\chi^{-1})''\chi'^2 + (\chi^{-1})'\tilde{F}_{0,2}\chi'^2])\\
& + \cancel{\frac{1}{2}(\chi^{-1})'''\chi'^3} + (\chi^{-1})''(\chi', \chi'')\\
& + \frac{1}{2}(\chi^{-1})''(\chi', \tilde{F}_{0,2}(\chi'^2)) + \frac{1}{2}(\chi^{-1})'\tilde{F}_{1,2}(\chi'^3) + (\chi^{-1})'\tilde{F}_{0,2}(\chi', \chi'')\\
& - \cancel{\frac{1}{2}(\chi^{-1})'''\chi'^3} - \frac{3}{2}(\chi^{-1})''(\tilde{F}_{0,2}(\chi'^2), \chi') - \frac{1}{2}(\chi^{-1})'\tilde{F}_{0,3}\chi'^3\\
=& \cancel{(\chi^{-1})''(\chi', \chi'(\chi^{-1})''\chi'^2)} + \underline{(\chi^{-1})''(\chi', \tilde{F}_{0,2}\chi'^2)}\\
& + \cancel{(\chi^{-1})'\tilde{F}_{0,2}(\chi', \chi'(\chi^{-1})''\chi'^2)} + (\chi^{-1})'\tilde{F}_{0,2}(\chi', \tilde{F}_{0,2}\chi'^2)\\
& + \cancel{(\chi^{-1})''(\chi', \chi'')}\\
& + \underline{\frac{1}{2}(\chi^{-1})''(\chi', \tilde{F}_{0,2}(\chi'^2))} + \frac{1}{2}(\chi^{-1})'\tilde{F}_{1,2}(\chi'^3) + \cancel{(\chi^{-1})'\tilde{F}_{0,2}(\chi', \chi'')}\\
& \underline{- \frac{3}{2}(\chi^{-1})''(\tilde{F}_{0,2}\chi'^2, \chi')} - \frac{1}{2}(\chi^{-1})'\tilde{F}_{0,3}\chi'^3\\
=& (\chi^{-1})'\tilde{F}_{0,2}(\chi', \tilde{F}_{0,2}\chi'^2) + \frac{1}{2}(\chi^{-1})'\tilde{F}_{1,2}(\chi'^3) - \frac{1}{2}(\chi^{-1})'\tilde{F}_{0,3}\chi'^3 = (\chi^{-1})'\tilde{P}(\chi'\,\cdot).
\end{aligned}
$$

[5]If one is confused by the multilinear map notation, assume that the manifold has dimension $n = 1$; then all the partial derivatives are scalars, and any term of the form "(an m-multilinear map) $(u_1, \ldots, u_m)$" is simply a product of scalars. One can already get a good "feel" for the subtlety of the derivation in the univariate case.

Above, we use also the identity $(\chi^{-1})''(\chi'^2) = -(\chi^{-1})'\chi''$. Since χ' is nonsingular, P vanishes if and only if $\tilde{P}$ vanishes. ∎

Now we know that P is a geometric invariant of the retraction map f; what we do not know yet is what it actually measures about f. From the two main results of this article (Theorems 5 and 8), it is tempting to speculate that P may in some sense measure how close f is to being the exponential map of a certain metric on the manifold. This is, of course, quite an immature guess at this point.

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