An improved Proximity $\Longrightarrow$ Smoothness theorem

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Abstract:

In this note, we point out that the proximity condition proposed in [8] can be weakened. We reprove the "Proximity ⇒ Smoothness" theorem under the so-called weak proximity condition. This improvement is embarrassingly trivial; and appears to be just an artifact of the authors overlooking a detail during the writing of [8]. But this raises a question: the unnecessarily strong proximity condition proposed in [8] was used in a number of subsequent papers and, according to this note the authors of these subsequent papers had worked unnecessarily hard to establish the strong proximity condition. If the original proximity condition is truly unnecessary, why would it persist in all these subsequent papers? Coupling the observation in this note and the main result in our recent work [1], we conclude that, under a natural compatibility condition, the strong and weak proximity conditions are *equivalent*. This equivalence is attributable to an algebraic structure of subdivision schemes.

The goal of this note is prove Theorem 0.7 below; it weakens the proximity condition in [8, Theorem 2.4] to a weaker proximity condition.

Throughout this section, we let $\mathbb{Z}^+ := \{0\} \cup \mathbb{N}$. For any sequence $x = (x_i)_i$ in an Euclidian space, let $|x|_\infty := \sup_i \|x_i\|_2$.

For any $M \subseteq \mathbb{R}^N$ and $\delta > 0$, let

$$\mathcal{X}_M := \left\{ x : \mathbb{Z} \to M \;\middle|\; |\Delta x|_\infty < \infty \right\} \text{ and } \mathcal{X}_{M,\delta} := \left\{ x : \mathbb{Z} \to M \;\middle|\; |\Delta x|_\infty < \delta \right\}.$$

For $j \in \mathbb{N}$, let

$$\Gamma_j := \left\{ \gamma = (\gamma_1, \cdots, \gamma_j) \;\middle|\; \gamma_i \in \mathbb{Z}^+,\ \sum_{i=1}^{j} \gamma_i \geqslant 2,\ \sum_{i=1}^{j} i\,\gamma_i = j+1 \right\}.$$

Note that $2 \leqslant |\gamma| := \gamma_1 + \cdots + \gamma_j \leqslant j+1$ for any $\gamma \in \Gamma_j$. Obviously, Γ_j is a finite set for every $j \in \mathbb{N}$. For any $x \in \mathcal{X}_M$, let

$$\Omega_j(x) := \sum_{\gamma \in \Gamma_j} \prod_{i=1}^{j} |\Delta^i x|_\infty^{\gamma_i}.$$

For example, $\Omega_1(x) = |\Delta x|_\infty^2$ and $\Omega_2(x) = |\Delta x|_\infty^3 + |\Delta x|_\infty |\Delta^2 x|_\infty$.

Since $|\Delta^k x|_\infty \leqslant 2|\Delta^{k-1} x|_\infty$ for any $k \in \mathbb{N}$, it follows that for any $j \in \mathbb{N}$, there exists a constant α_j such that for any $x \in \mathcal{X}_{M,\delta}$,

$$\Omega_{j+1}(x) \leqslant \alpha_j \Omega_j(x). \tag{0.1}$$

Lemma 0.1. Let $S : \mathcal{X}_{M,\delta} \to \mathcal{X}_M$. Let $\bar{S}$ be a convergent linear subdivision operator with dilation factor D. Suppose there exist $\beta > 1$, $C_1 > 0$ such that

$$|Sx - \bar{S}x|_\infty \leqslant C_1 |\Delta x|_\infty^\beta, \quad \forall\, x \in \mathcal{X}_{M,\delta},$$

then there exist $\delta' > 0$, $C > 0$ and $\alpha > 0$ such that $S^j x$ is well-defined and $|\Delta S^j x|_\infty \leqslant C D^{-j\alpha} |\Delta x|_\infty$ for any $j \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$.

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Proof: Without loss of generality, we can assume $\delta < 1$.
(i) We prove that for each $J \in \mathbb{N}$, there exists a $\delta_J > 0$ such that $S^J x$ is well-defined for any $x \in \mathcal{X}_{M,\delta_J}$. It is equivalent to showing that there exists a $\delta_J > 0$ such that $|\Delta S^j x|_\infty < \delta$ for $j = 0, \cdots, J-1$ and $x \in \mathcal{X}_{M,\delta_J}$.

We use induction. When $J = 1$, we can choose $\delta_J = \delta$. Suppose $|\Delta S^j x|_\infty < \delta$ for $j = 0, \cdots, J-1$ and $x \in \mathcal{X}_{M,\delta_J}$.

Since $\bar{S}$ is convergent, there exist $\gamma > 0$ and $\bar{C} > 0$ such that

$$|\Delta \bar{S}^j x|_\infty \leqslant \bar{C} D^{-j\gamma} |\Delta x|_\infty$$

for any $x \in \mathcal{X}_M$ and $j \in \mathbb{N}$.

It follows that

$$\begin{aligned} |\Delta S^J x|_\infty &\leqslant |\Delta \bar{S} S^{J-1} x|_\infty + |\Delta S^J x - \Delta \bar{S} S^{J-1} x|_\infty \\ &\leqslant \bar{C} D^{-\gamma} |\Delta S^{J-1} x|_\infty + 2|S^J x - \bar{S} S^{J-1} x|_\infty \\ &\leqslant \bar{C} D^{-\gamma} |\Delta S^{J-1} x|_\infty + 2C_1 |\Delta S^{J-1} x|_\infty^\beta \\ &\leqslant (\bar{C} D^{-\gamma} + 2C_1) |\Delta S^{J-1} x|_\infty \\ &\leqslant \cdots \\ &\leqslant (\bar{C} D^{-\gamma} + 2C_1)^J |\Delta x|_\infty. \end{aligned} \tag{0.2}$$

Therefore we can choose

$$\delta_{J+1} := \min \left\{ \delta_J, \frac{\delta}{(\bar{C} D^{-\gamma} + 2C_1)^J} \right\}.$$

(ii) We prove that for each $K \in \mathbb{N}$, there exists a $C_K > 0$ such that

$$|S^K x - \bar{S}^K x|_\infty \leqslant C_K |\Delta x|_\infty^\beta$$

whenever $S^K x$ is well-defined.

We use induction. The statement is true for $K = 1$ by assumption. Suppose that it is true for some $K \in \mathbb{N}$. Then it follows from (0.2) that

$$\begin{aligned} |S^{K+1} x - \bar{S}^{K+1} x|_\infty &\leqslant |S^{K+1} x - \bar{S} S^K x|_\infty + |\bar{S} S^K x - \bar{S}^{K+1} x|_\infty \\ &\leqslant C_1 |\Delta S^K x|_\infty^\beta + |\bar{S}|_\infty C_K |\Delta x|_\infty^\beta \\ &\leqslant C_1 (\bar{C} D^{-\gamma} + 2C_1)^{K\beta} |\Delta x|_\infty^\beta + |\bar{S}|_\infty C_K |\Delta x|_\infty^\beta \\ &= (C_1 (\bar{C} D^{-\gamma} + 2C_1)^{K\beta} + |\bar{S}|_\infty C_K) |\Delta x|_\infty^\beta. \end{aligned}$$

Therefore we can choose $C_{K+1} := C_1 (\bar{C} D^{-\gamma} + 2C_1)^{K\beta} + |\bar{S}|_\infty C_K$.
(iii) We prove that there exist $C > 0$, $\alpha > 0$ and $\delta_0 > 0$ such that

$$|\Delta S^j x|_\infty \leqslant C D^{-j\alpha} |\Delta x|_\infty$$

whenever $S^j x$ is well-defined and $|\Delta x|_\infty < \delta_0$.

Choose any $0 < \alpha < \gamma$ and $K \in \mathbb{N}$ such that $K > \frac{\log_D \bar{C}}{\gamma - \alpha}$. Then $\bar{C} D^{-K\gamma} < D^{-K\alpha}$. It follows from (i)(ii) that there exists $C_K > 0$ and $\delta_K > 0$ such that $S^K x$ exists for $x \in \mathcal{X}_{M,\delta_K}$ and

$$|S^K x - \bar{S}^K x|_\infty \leqslant C_K |\Delta x|_\infty^\beta.$$

Let $\delta'_K := \min \left\{ \delta_K, \left(\frac{D^{-K\alpha} - \bar{C} D^{-K\gamma}}{2C_K} \right)^{\frac{1}{\beta - 1}} \right\}$. Then for any $x \in \mathcal{X}_{M,\delta'_K}$, we have

$$\begin{aligned} |\Delta S^K x|_\infty &\leqslant |\Delta \bar{S}^K x|_\infty + |\Delta S^K x - \Delta \bar{S}^K x|_\infty \\ &\leqslant \bar{C} D^{-K\gamma} |\Delta x|_\infty + 2C_K |\Delta x|_\infty^\beta \\ &= (\bar{C} D^{-K\gamma} + 2C_K |\Delta x|_\infty^{\beta - 1}) |\Delta x|_\infty \\ &\leqslant D^{-K\alpha} |\Delta x|_\infty. \end{aligned} \tag{0.3}$$

It follows from (0.2) that whenever $S^j x$ is well-defined, we have

$$|\Delta S^j x|_\infty \leqslant \tilde{C}^j |\Delta x|_\infty, \tag{0.4}$$

where $\tilde{C} = \bar{C} D^{-\gamma} + 2C_1$.

Let $\delta_0 := \tilde{C}^{-K} \delta'_K$. Then it follows from (0.3)(0.4) that whenever $S^j x$ is well-defined and $|\Delta x|_\infty < \delta_0$,

$$|\Delta S^j x|_\infty = |\Delta S^{\ell K + r} x|_\infty \leqslant D^{-\ell K \alpha} |\Delta S^r x|_\infty \leqslant D^{-\ell K \alpha} \tilde{C}^r |\Delta x|_\infty = D^{-j\alpha} D^{r\alpha} \tilde{C}^r |\Delta x|_\infty \leqslant C D^{-j\alpha} |\Delta x|_\infty,$$

where $C := \max_{0 \leqslant r < K} D^{r\alpha} \tilde{C}^r$.

(iv) Last, we prove that there exists a $\delta' > 0$ such that $S^j x$ is well-defined for any $j \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$.

Let $\delta' := \min(\delta, \delta/C, \delta_0)$ where C and δ_0 are from (iii). Then we use induction to prove that $S^j x$ is well-defined for any $j \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$. When $j = 1$, this is obviously true. Suppose $S^j x$ is well-defined for any $x \in \mathcal{X}_{M,\delta'}$. Then it follows from (iii) that

$$|\Delta S^j x|_\infty \leqslant C D^{-j\alpha} |\Delta x|_\infty < C |\Delta x|_\infty \leqslant \delta,$$

for any $x \in \mathcal{X}_{M,\delta'}$. Therefore $S^{j+1} x$ is well-defined for any $x \in \mathcal{X}_{M,\delta'}$.

Combining (iii) and (iv), we have

$$|\Delta S^j x|_\infty \leqslant C D^{-j\alpha} |\Delta x|_\infty$$

for any $j \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$. ■

For any sequence $x : \mathbb{Z} \to \mathbb{R}^N$, $n \in \mathbb{Z}^+$, and $D > 1, D \in \mathbb{N}$, we define $\mathcal{F}_D^n(x)$ to be the piecewise linear function with $\mathcal{F}_D^n(x)(iD^{-n}) = x_i$. For fixed D and n, $\mathcal{F}_D^n$ can be viewed as a linear map from the space of sequences x to the space of piecewise linear functions; moreover, we have

$$|\mathcal{F}_D^n(x) - \mathcal{F}_D^n(\tilde{x})|_\infty = |x - \tilde{x}|_\infty, \quad \forall\, x, \tilde{x}. \tag{0.5}$$

For a subdivision operator $S : \mathcal{X}_{M,\delta} \to \mathcal{X}_M$ with dilation factor D, if there exists $\delta' > 0$ such that $\mathcal{F}_D^n(S^n x)$ converges to a C^k ($k \in \mathbb{Z}^+$) function as $n \to \infty$ for any $x \in \mathcal{X}_{M,\delta'}$, then we say S is C^k.

To prove S is C^k ($k \in \mathbb{Z}^+$), it suffices to show that there exists a $\delta' > 0$ such that for any $x \in \mathcal{X}_{M,\delta'}$ and $j = 0, 1, \cdots, k$, $\mathcal{F}_D^n(D^{jn} \Delta^j S^n x)$ converges uniformly as $n \to \infty$. See [3, Theorem 4.1].

We need the following lemma. Its proof can be found in [2] or [6, Lemma 1].

Lemma 0.2. Let $\bar{S}$ be a convergent linear subdivision operator with dilation factor D. Then there exists a constant $\bar{C} > 0$ such that for any sequence x and $n \in \mathbb{N}$,

$$|\mathcal{F}_D^n(\bar{S} x) - \mathcal{F}_D^{n-1}(x)|_\infty \leqslant \bar{C} |\Delta x|_\infty.$$

We first prove the following basic "proximity ⇒ continuity" result.

Theorem 0.3. Let $S : \mathcal{X}_{M,\delta} \to \mathcal{X}_M$ be a subdivision operator and $\bar{S}$ be a convergent linear subdivision operator. Suppose S and $\bar{S}$ have the same dilation factor and there exists $C_1 > 0$ and $\beta > 1$ such that

$$|Sx - \bar{S}x|_\infty \leqslant C_1 |\Delta x|_\infty^\beta, \quad \forall\, x \in \mathcal{X}_{M,\delta}. \tag{0.6}$$

Then S is C^0.

Proof: Suppose S and $\bar{S}$ have dilation factor D. It follows from Lemma 0.1 that there exists $C > 0$, $\alpha > 0$ and $\delta' > 0$ such that $S^n x$ is well-defined and $|\Delta S^n x|_\infty \leqslant C D^{-n\alpha} |\Delta x|_\infty$ for any $n \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$. Combining with Lemma 0.2, (0.5) and (0.6), we have for any $n \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$,

$$\begin{aligned}
|\mathcal{F}_D^n(S^n x) - \mathcal{F}_D^{n+1}(S^{n+1} x)|_\infty &\leqslant |\mathcal{F}_D^n(S^n x) - \mathcal{F}_D^{n+1}(\bar{S} S^n x)|_\infty + |\mathcal{F}_D^{n+1}(\bar{S} S^n x) - \mathcal{F}_D^{n+1}(S^{n+1} x)|_\infty \\
&\leqslant \bar{C} |\Delta S^n x|_\infty + |S^{n+1} x - \bar{S} S^n x|_\infty \\
&\leqslant \bar{C} |\Delta S^n x|_\infty + C_1 |\Delta S^n x|_\infty^\beta \\
&\leqslant \bar{C} C D^{-n\alpha} |\Delta x|_\infty + C_1 C^\beta D^{-n\alpha\beta} |\Delta x|_\infty^\beta \\
&\leqslant \bar{C} C D^{-n\alpha} \delta' + C_1 C^\beta D^{-n\alpha\beta} \delta'^\beta.
\end{aligned}$$

Therefore $\mathcal{F}_D^n(S^n x)$ $(n \in \mathbb{N})$ is a Cauchy sequence. Note $\mathcal{F}_D^n(S^n x)$ is a C^0 function for any $n \in \mathbb{N}$. Hence $\mathcal{F}_D^n(S^n x)$ converges to a C^0 function as $n \to \infty$. This means S is C^0. ∎

The final goal of this section is to extend Theorem 0.3 to higher order smoothness. To prove this theorem we need to first develop a few auxiliary lemmas.

Lemma 0.4. *Let S be a subdivision operator defined on $\mathcal{X}_{M,\delta}$ and $\bar{S}$ be a linear subdivision operator. Both of them have the same dilation factor D. Suppose there exist $\mu_1, \cdots, \mu_{k+1} \in [0,1)$ such that*

$$|\Delta^j \bar{S} x|_\infty \leqslant D^{-j+\mu_j} |\Delta^j x|_\infty, \quad \forall x, j = 1, \cdots, k+1,$$

and

$$\mu_j < \frac{\mu_{j+1}}{j+1}, \quad j = 1, \cdots, k.$$

If there exists $C > 0$ and $\beta > 1$ such that for any $x \in \mathcal{X}_{M,\delta}$ and $j = 1, \cdots, k$

$$|Sx - \bar{S}x|_\infty \leqslant C |\Delta x|_\infty^\beta$$

$$|\Delta^j S x - \Delta^j \bar{S} x|_\infty \leqslant C \Omega_j(x).$$

Then for any $0 < \epsilon < \min_{1 \leqslant j \leqslant k} \left(\frac{\mu_{j+1}}{j+1} - \mu_j \right)$, there exist $0 < \delta' \leqslant \delta$ and polynomials $P_2, \cdots, P_{k+1}$ such that for any $n \in \mathbb{N}$ and any $x \in \mathcal{X}_{M,\delta'}$

$$|\Delta S^n x|_\infty \leqslant D^{(-1+\mu_1+\epsilon)n} |\Delta x|_\infty,$$
$$|\Delta^j S^n x|_\infty \leqslant D^{(-j+\mu_j)n} P_j(n) |\Delta x|_\infty, \quad j = 2, \cdots, k+1.$$

Proof: Since $\Omega_1(x) = |\Delta x|_\infty^2$, $j = 1$ case has been proved in [7, Theorem 2], i.e. for any $0 < \epsilon < \min_{1 \leqslant j \leqslant k} \left(\frac{\mu_{j+1}}{j+1} - \mu_j \right)$, there exists $0 < \delta' \leqslant \delta$ such that for any $x \in \mathcal{X}_{M,\delta'}$ and any $n \in \mathbb{N}$,

$$|\Delta S^n x|_\infty \leqslant D^{(-1+\mu_1+\epsilon)n} |\Delta x|_\infty.$$

We proceed by induction in j. Now assume the result is true for $1, \cdots, j$ $(j \leqslant k)$. Now for any $n \in \mathbb{N}$,

$$|\Delta^{j+1} S^n x|_\infty \leqslant |\Delta^{j+1} \bar{S} S^{n-1} x|_\infty + |\Delta^{j+1} S^n x - \Delta^{j+1} \bar{S} S^{n-1} x|_\infty.$$

Since for any sequences y, z, we have $|\Delta y - \Delta z|_\infty = |\Delta(y-z)|_\infty \leqslant 2|y-z|_\infty$. It follows that

$$\begin{aligned} |\Delta^{j+1} S^n x|_\infty &\leqslant |\Delta^{j+1} \bar{S} S^{n-1} x|_\infty + 2|\Delta^j S^n x - \Delta^j \bar{S} S^{n-1} x|_\infty \\ &\leqslant D^{-j-1+\mu_{j+1}} |\Delta^{j+1} S^{n-1} x|_\infty + 2C\Omega_j(S^{n-1} x). \end{aligned}$$

Denote $d_n := |\Delta^{j+1} S^n x|_\infty$. Then

$$\begin{aligned} d_n &\leqslant D^{-j-1+\mu_{j+1}} d_{n-1} + 2C\Omega_j(S^{n-1}x) \\ &\leqslant \cdots \\ &\leqslant D^{(-j-1+\mu_{j+1})n} d_0 + 2C \sum_{i=0}^{n-1} D^{(-j-1+\mu_{j+1})(n-1-i)} \Omega_j(S^i x) \\ &\leqslant D^{(-j-1+\mu_{j+1})n} \left(d_0 + 2CD^{j+1} \sum_{i=0}^{n-1} D^{i(j+1-\mu_{j+1})} \Omega_j(S^i x) \right). \end{aligned}$$

Since

$$\begin{aligned}
\Omega_j(S^i x) &= \sum_{\gamma\in\Gamma_j} |\Delta S^i x|_\infty^{\gamma_1}\cdots|\Delta^j S^i x|_\infty^{\gamma_j} \\
&\leqslant \sum_{\gamma\in\Gamma_j} \left((D^{-1+\mu_1+\epsilon})^i|\Delta x|_\infty\right)^{\gamma_1}\cdots\left((D^{-j+\mu_j})^i P_j(i)|\Delta x|_\infty\right)^{\gamma_j} \\
&= \sum_{\gamma\in\Gamma_j} |\Delta x|_\infty^{|\gamma|} P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j} D^{i(\mu_1\gamma_1+\cdots+\mu_j\gamma_j+\epsilon\gamma_1-j-1)} \\
&\leqslant \sum_{\gamma\in\Gamma_j} |\Delta x|_\infty^{|\gamma|} P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j} D^{i(\mu_j|\gamma|+\epsilon|\gamma|-j-1)} \\
&\leqslant \sum_{\gamma\in\Gamma_j} |\Delta x|_\infty^{|\gamma|} P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j} D^{i(j+1)(\mu_j+\epsilon-1)} \\
&\leqslant |\Delta x|_\infty \sum_{\gamma\in\Gamma_j} \delta'^{|\gamma|-1} P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j} D^{i(j+1)(\mu_j+\epsilon-1)}
\end{aligned}$$

and

$$d_0 = |\Delta^{j+1}x|_\infty \leqslant 2^j|\Delta x|_\infty.$$

It follows that

$$\begin{aligned}
d_n &\leqslant D^{(-j-1+\mu_{j+1})n}\left(2^j + 2CD^{j+1}\sum_{i=0}^{n-1} D^{i(j+1-\mu_{j+1})}\sum_{\gamma\in\Gamma_j}\delta'^{|\gamma|-1}P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j}D^{i(j+1)(\mu_j+\epsilon-1)}\right)|\Delta x|_\infty \\
&\leqslant D^{(-j-1+\mu_{j+1})n}\left(2^j + 2CD^{j+1}\sum_{i=0}^{n-1}\sum_{\gamma\in\Gamma_j}\delta'^{|\gamma|-1}P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j}D^{i(j+1)(\mu_j+\epsilon-\frac{\mu_{j+1}}{j+1})}\right)|\Delta x|_\infty \\
&\leqslant D^{(-j-1+\mu_{j+1})n}\left(2^j + 2CD^{j+1}\sum_{i=0}^{n-1}\sum_{\gamma\in\Gamma_j}\delta'^{|\gamma|-1}P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j}\right)|\Delta x|_\infty,
\end{aligned}$$

where the last inequality is due to the fact that $\mu_j + \epsilon - \frac{\mu_{j+1}}{j+1} < 0$. Define

$$P_{j+1}(n) = 2^j + 2CD^{j+1}\sum_{i=0}^{n-1}\sum_{\gamma\in\Gamma_j}\delta'^{|\gamma|-1}P_2(i)^{\gamma_1}\cdots P_j(i)^{\gamma_j}.$$

Then

$$|\Delta^{j+1}S^n x|_\infty = d_n \leqslant D^{(-j-1+\mu_{j+1})n}P_{j+1}(n)|\Delta x|_\infty.$$

Hence the lemma is proved. ■

Lemma 0.5. *Let S be a subdivision operator defined on $\mathcal{X}_{M,\delta'}$ and $\bar{S}$ be a linear subdivision operator such that its derived subdivision operators $\bar{S}_1,\cdots,\bar{S}_k$ $(k\geqslant 1)$ all exist. S and $\bar{S}$ have the same dilation factor D. Suppose there exists $C>0$ and $\beta>1$ such that for any $x\in\mathcal{X}_{M,\delta'}$ and $j=1,\cdots,k$*

$$|Sx-\bar{S}x|_\infty \leqslant C|\Delta x|_\infty^\beta$$

$$|\Delta^j Sx - \Delta^j\bar{S}x|_\infty \leqslant C\Omega_j(x).$$

If there exist $\mu_1,\cdots,\mu_{k+1}\in[0,1)$ satisfying

$$\mu_j < \frac{\mu_{j+1}}{j+1},\quad j=1,\cdots,k,$$

and polynomials $P_1,\cdots,P_{k+1}$ such that for any $n\in\mathbb{N}$ and any $x\in\mathcal{X}_{M,\delta'}$

$$|\Delta^j S^n x|_\infty \leqslant D^{(-j+\mu_j)n}P_j(n)|\Delta x|_\infty,\quad j=1,\cdots,k+1.$$

Then S is C^k.

Proof: It follows from Theorem 0.3 that S is continuous. We only need to show for $j = 1, \cdots, k$, $\mathcal{F}_D^n(D^{jn}\Delta^j S^n x)$ converges uniformly as $n \to \infty$ for all $x \in \mathcal{X}_{M,\delta'}$. Consider

$$\begin{aligned}
&|\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j S^{n+1}x) - \mathcal{F}_D^n(D^{jn}\Delta^j S^n x)|_\infty \\
\leqslant &|\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j S^{n+1}x) - \mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j \bar{S} S^n x)|_\infty + |\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j \bar{S} S^n x) - \mathcal{F}_D^n(D^{jn}\Delta^j S^n x)|_\infty.
\end{aligned}$$

Using (0.5), we have

$$\begin{aligned}
&|\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j S^{n+1}x) - \mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j \bar{S} S^n x)|_\infty \\
=&|D^{j(n+1)}\Delta^j S^{n+1}x - D^{j(n+1)}\Delta^j \bar{S} S^n x|_\infty \\
\leqslant &CD^{j(n+1)}\Omega_j(S^n x) \\
=&CD^{j(n+1)}\sum_{\gamma\in\Gamma_j}|\Delta S^n x|_\infty^{\gamma_1}\cdots|\Delta^j S^n x|_\infty^{\gamma_j} \\
\leqslant &CD^{j(n+1)}\sum_{\gamma\in\Gamma_j}\left(D^{(-1+\mu_1)n}P_1(n)|\Delta x|_\infty\right)^{\gamma_1}\cdots\left(D^{(-j+\mu_j)n}P_j(n)|\Delta x|_\infty\right)^{\gamma_j} \\
=&CD^{j(n+1)}\sum_{\gamma\in\Gamma_j}D^{n(\mu_1\gamma_1+\cdots+\mu_j\gamma_j-j-1)}P_1(n)^{\gamma_1}\cdots P_j(n)^{\gamma_j}|\Delta x|_\infty^{|\gamma|} \\
\leqslant &CD^{j(n+1)}\sum_{\gamma\in\Gamma_j}D^{n(\mu_j(j+1)-j-1)}P_1(n)^{\gamma_1}\cdots P_j(n)^{\gamma_j}\delta'^{|\gamma|} \\
\leqslant &CD^{j(n+1)}\sum_{\gamma\in\Gamma_j}D^{n(\mu_{j+1}-j-1)}P_1(n)^{\gamma_1}\cdots P_j(n)^{\gamma_j}\delta'^{|\gamma|} \\
=&D^{n(\mu_{j+1}-1)}\left(CD^j\sum_{\gamma\in\Gamma_j}P_1(n)^{\gamma_1}\cdots P_j(n)^{\gamma_j}\delta'^{|\gamma|}\right).
\end{aligned}$$

Since $\mu_{j+1}-1<0$ and $P_1, \cdots, P_j$ are polynomials, it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}|\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j S^{n+1}x) - \mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j \bar{S} S^n x)|_\infty \\
\leqslant &\sum_{n=1}^{\infty}D^{n(\mu_{j+1}-1)}\left(CD^j\sum_{\gamma\in\Gamma_j}P_1(n)^{\gamma_1}\cdots P_j(n)^{\gamma_j}\delta'^{|\gamma|}\right)<\infty.
\end{aligned}$$

Using Lemma 0.2, we have

$$\begin{aligned}
&|\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j \bar{S} S^n x) - \mathcal{F}_D^n(D^{jn}\Delta^j S^n x)|_\infty \\
=&D^{jn}|\mathcal{F}_D^{n+1}(D^j\Delta^j \bar{S} S^n x) - \mathcal{F}_D^n(\Delta^j S^n x)|_\infty \\
=&D^{jn}|\mathcal{F}_D^{n+1}(\bar{S}_j\Delta^j S^n x) - \mathcal{F}_D^n(\Delta^j S^n x)|_\infty \\
\leqslant &\bar{C}D^{jn}|\Delta^{j+1}S^n x|_\infty \\
\leqslant &\bar{C}D^{jn}D^{(-j-1+\mu_{j+1})n}P_{j+1}(n)|\Delta x|_\infty \\
\leqslant &\bar{C}D^{n(\mu_{j+1}-1)}P_{j+1}(n)\delta'.
\end{aligned}$$

Since $\mu_{j+1}-1<0$ and P_{j+1} is a polynomial, it follows that

$$\sum_{n=1}^{\infty}|\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j \bar{S} S^n x) - \mathcal{F}_D^n(D^{jn}\Delta^j S^n x)|_\infty \leqslant \sum_{n=1}^{\infty}\bar{C}D^{n(\mu_{j+1}-1)}P_{j+1}(n)\delta' < \infty.$$

Combining all the above, we obtain

$$\sum_{n=1}^{\infty}|\mathcal{F}_D^{n+1}(D^{j(n+1)}\Delta^j S^{n+1}x) - \mathcal{F}_D^n(D^{jn}\Delta^j S^n x)|_\infty < \infty.$$

Hence $\mathcal{F}_D^n(D^{jn}\Delta^j S^n x)$ is a Cauchy sequence with respect to $|\cdot|_\infty$. Therefore $\mathcal{F}_D^n(D^{jn}\Delta^j S^n x)$ converges uniformly as $n \to \infty$. ■

Lemma 0.6. *Let S be a subdivision operator defined on $\mathcal{X}_{M,\delta}$ and $\bar{S}$ be a linear subdivision operator such that its derived subdivision operators $\bar{S}_1, \cdots, \bar{S}_k$ all exist. Suppose S and $\bar{S}$ have the same dilation factor. If there exists $C_1 > 0$ such that for any $x \in \mathcal{X}_{M,\delta}$ and $j = 1, \cdots, k$*

$$|\Delta^j Sx - \Delta^j \bar{S}x|_\infty \leqslant C_1 \Omega_j(x).$$

Then for any $n \in \mathbb{N}$, there exists $C_n > 0$ such that for any $x \in \mathcal{X}_{M,\delta}$ and $j = 1, \cdots, k$

$$|\Delta^j S^n x - \Delta^j \bar{S}^n x|_\infty \leqslant C_n \Omega_j(x).$$

Proof: Suppose the dilation factor of S and $\bar{S}$ is D. Then $\bar{S}_j \Delta^j = D^j \Delta^j \bar{S}$ for $j = 1, \cdots, k$. Hence

$$\Delta^j \bar{S}^n = D^{-nj} \bar{S}_j^n \Delta^j, \quad \forall n \in \mathbb{N},\ j = 1, \cdots, k.$$

We use induction to prove the result. When $n = 1$, the result is true by assumption. Now assume the result is true for $n-1$. Then for $j = 1, \cdots, k$

$$\begin{aligned}
|\Delta^j S^n x - \Delta^j \bar{S}^n x|_\infty &\leqslant |\Delta^j S^n x - \Delta^j \bar{S} S^{n-1} x|_\infty + |\Delta^j \bar{S} S^{n-1} x - \Delta^j \bar{S}^n x|_\infty \\
&\leqslant C_{n-1} \Omega_j(S^{n-1}x) + D^{-j} |\bar{S}_j \Delta^j S^{n-1} x - \bar{S}_j \Delta^j \bar{S}^{n-1} x|_\infty \\
&\leqslant C_{n-1} \Omega_j(S^{n-1}x) + D^{-j} |\bar{S}_j|_\infty C_{n-1} \Omega_j(x).
\end{aligned}$$

Since for $j = 1, \cdots, k$,

$$\begin{aligned}
|\Delta^j S^{n-1} x|_\infty &\leqslant |\Delta^j \bar{S}^{n-1} x|_\infty + |\Delta^j S^{n-1} x - \Delta^j \bar{S}^{n-1} x|_\infty \\
&\leqslant D^{-(n-1)j} |\bar{S}_j^{n-1} \Delta^j x|_\infty + C_{n-1} \Omega_j(x) \\
&\leqslant D^{-(n-1)j} |\bar{S}_j^{n-1}|_\infty |\Delta^j x|_\infty + C_{n-1} \Omega_j(x).
\end{aligned}$$

It follows that

$$\begin{aligned}
\Omega_j(S^{n-1}x) &= \sum_{\gamma \in \Gamma_j} |\Delta S^{n-1} x|_\infty^{\gamma_1} \cdots |\Delta^j S^{n-1} x|_\infty^{\gamma_j} \\
&\leqslant \sum_{\gamma \in \Gamma_j} \Big(D^{-(n-1)} |\bar{S}_1^{n-1}|_\infty |\Delta x|_\infty + C_{n-1} \Omega_1(x) \Big)^{\gamma_1} \cdots \Big(D^{-(n-1)j} |\bar{S}_j^{n-1}|_\infty |\Delta^j x|_\infty + C_{n-1} \Omega_j(x) \Big)^{\gamma_j}.
\end{aligned}$$

It is easy to verify that for any $\gamma \in \Gamma_j$, there exists $C_{\gamma,n-1} > 0$ such that

$$\Big(D^{-(n-1)} |\bar{S}_1^{n-1}|_\infty |\Delta x|_\infty + C_{n-1} \Omega_1(x) \Big)^{\gamma_1} \cdots \Big(D^{-(n-1)j} |\bar{S}_j^{n-1}|_\infty |\Delta^j x|_\infty + C_{n-1} \Omega_j(x) \Big)^{\gamma_j} \leqslant C_{\gamma,n-1} \Omega_j(x),$$

where $C_{\gamma,n-1}$ depends on D, γ, C_{n-1} and $|\bar{S}_1^{n-1}|_\infty, \cdots, |\bar{S}_j^{n-1}|_\infty$. Therefore

$$\Omega_j(S^{n-1}x) \leqslant \left(\sum_{\gamma \in \Gamma_j} C_{\gamma,n-1} \right) \Omega_j(x).$$

Hence

$$\begin{aligned}
|\Delta^j S^n x - \Delta^j \bar{S}^n x|_\infty &\leqslant C_{n-1} \left(\sum_{\gamma \in \Gamma_j} C_{\gamma,n-1} \right) \Omega_j(x) + D^{-j} |\bar{S}_{j-1}|_\infty C_{n-1} \Omega_j(x) \\
&\leqslant C_n \Omega_j(x),
\end{aligned}$$

where

$$C_n = C_{n-1} \max_{1 \leqslant j \leqslant k} \left(\sum_{\gamma \in \Gamma_j} C_{\gamma,n-1} + D^{-j} |\bar{S}_{j-1}|_\infty \right).$$

■

Theorem 0.7. Let S be a subdivision operator defined on $\mathcal{X}_{M,\delta}$ and $\bar{S}$ be a C^k ($k \geqslant 1$) linear L_∞-stable subdivision operator. Suppose S and $\bar{S}$ have the same dilation factor. If there exists $C_1 > 0$ and $\beta > 1$ such that for any $x \in \mathcal{X}_{M,\delta}$ and $j = 1, \cdots, k$

$$|Sx - \bar{S}x|_\infty \leqslant C_1 |\Delta x|_\infty^\beta$$

$$|\Delta^j Sx - \Delta^j \bar{S}x|_\infty \leqslant C_1 \Omega_j(x).$$

Then S is C^k.

Proof: Suppose the dilation factor of S and $\bar{S}$ is D. Since $\bar{S}$ is C^k ($k \geqslant 1$), linear and L_∞-stable, it follows (e.g. [5, 4]) that there exist $\bar{C}_1, \cdots, \bar{C}_{k+1} \geqslant 1$ and $\mu \in [0, 1)$ such that

$$|\Delta^j \bar{S}^n x|_\infty \leqslant \bar{C}_j D^{-jn} |\Delta^j x|_\infty, \quad j = 1, \cdots, k; \tag{0.7}$$

$$|\Delta^{k+1} \bar{S}^n x|_\infty \leqslant \bar{C}_{k+1} D^{(-j-1+\mu)n} |\Delta^{k+1} x|_\infty. \tag{0.8}$$

We can artificially modify the above constants such that

$$\bar{C}_j^{j+1} < \bar{C}_{j+1}, \quad j = 1, \cdots, k. \tag{0.9}$$

Since $\mu < 1$, there exists $m \in \mathbb{N}$ such that

$$\bar{\mu}_{k+1} := \mu + \frac{\log_D \bar{C}_{k+1}}{m} < 1.$$

Define $\bar{\mu}_j = \frac{\log_D \bar{C}_j}{m}$ for $j = 1, \cdots, k$. Then it follows from (0.9) that

$$\bar{\mu}_j < \frac{\bar{\mu}_{j+1}}{j+1}, \quad j = 1, \cdots, k.$$

Let $T := S^m$ and $\bar{T} := \bar{S}^m$ be two new subdivision operators. Then both of them have dilation factor $\tilde{D} := D^m$ and $\bar{T}$ is C^k, linear and L_∞-stable. Hence $\bar{T}$ has derived subdivision operators $\bar{T}_1, \cdots, \bar{T}_k$. It follows from part(ii) of Lemma 0.1's proof and Lemma 0.6 that there exists $C_m > 0$ such that $\forall x \in \mathcal{X}_{M,\delta}$, we have

$$|Tx - \bar{T}x|_\infty \leqslant C_m |\Delta x|_\infty^\beta$$

$$|\Delta^j Tx - \Delta^j \bar{T}x|_\infty \leqslant C_m \Omega_j(x), \quad j = 1, \cdots, k.$$

It follows from (0.7)(0.8) that

$$|\Delta^j \bar{T}x|_\infty \leqslant \tilde{D}^{-j+\bar{\mu}_j} |\Delta^j x|_\infty, \quad j = 1, \cdots, k+1.$$

Choose

$$0 < \epsilon < \min\left(1 - \bar{\mu}_1, \min_{1 \leqslant j \leqslant k} \left(\frac{\bar{\mu}_{j+1}}{j+1} - \bar{\mu}_j\right)\right).$$

Define $\mu_1 = \bar{\mu}_1 + \epsilon$, $\mu_2 = \bar{\mu}_2, \cdots, \mu_{k+1} = \bar{\mu}_{k+1}$. Then $\mu_j \in [0, 1)$ for $j = 1, \cdots, k+1$ and

$$\mu_j < \frac{\mu_{j+1}}{j+1}, \quad j = 1, \cdots, k.$$

It follows from Lemma 0.4 that there exist $0 < \delta' \leqslant \delta$ and polynomials $P_1, \cdots, P_{k+1}$ such that for any $n \in \mathbb{N}$ and $x \in \mathcal{X}_{M,\delta'}$,

$$|\Delta^j T^n x|_\infty \leqslant \tilde{D}^{(-j+\mu_j)n} P_j(n) |\Delta x|_\infty, \quad j = 1, \cdots, k+1.$$

It follows from Lemma 0.5 that $T = S^m$ is C^k. Therefore S is C^k. ∎

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