Face-based Hermite Subdivision Schemes

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Abstract

Interpolatory and non-interpolatory multivariate Hermite type subdivision schemes are introduced in [8, 7]. In their applications in free-form surfaces, *symmetry* properties play a fundamental role: one can essentially argue that a subdivision scheme without a symmetry property simply *cannot* be used for the purpose of modelling free-form surfaces. The symmetry properties defined in the article [8] are formulated based on an underlying conception that Hermite data produced by the subdivision process is attached exactly to the *vertices* of the subsequently refined tessellations of the Euclidean space. As such, certain interesting possibilities of symmetric Hermite subdivision schemes are disallowed under our vertex-based symmetry definition. In this article, we formulate new symmetry conditions based on the conception that Hermite data produced in the subdivision process is attached to the *faces* instead of *vertices* of the subsequently refined tessellations. New examples of symmetric faced-based schemes are then constructed.

Similar to our earlier work in vertex-based interpolatory and non-interpolatory Hermite subdivision schemes, a key step in our analysis is that we make use of the *strong convergence theory* of refinement equation to convert a *prescribed geometric condition* on the subdivision scheme – namely, the subdivision scheme is of Hermite type – to an *algebraic condition* on the subdivision mask. Our quest for face-based schemes in this article leads also to a refined result in this direction.

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1 Introduction

Subdivision schemes are used in both curve and surface design as well as wavelet construction. Because of the two different applications, there comes also a certain degree of confusion in what subdivision is supposed to mean even in the most standard *linear, stationary and shift-invariant* case:

- **[Geo]** To a number of geometric modelling and computer graphics scientists, subdivision, in their so-called *regular* setting, typically consists of the following 4 components:

  [T1] a choice of isohedral tiling of $\mathbb{R}^s$;

  [T2] a *refinement rule* that is used to transform the tiling to a *similar* tiling but with a smaller tile size; this refinement rule is used repeatedly in the subdivision process to create finer and finer tilings of $\mathbb{R}^s$;

  [G1] a specification of *how* data (measuring geometric positions) is attached to the individual tiles; in 2-D this comes with the choice of attaching positional data to vertex and/or edge and/or face;

  [G2] a fixed linear rule for how data attached to any specific tile is determined from the data attached to the coarser scale tiles.

  [T1]-[T2] are typically referred to as the "topological" part of the subdivision scheme. So far the main application of subdivision is in surface modelling, in which $s = 2$ and the isohedral tiling is usually based on equilateral triangles or squares.

  (In the setting of free-form subdivision surfaces, the above is the so-called regular part of a subdivision scheme, one needs also extraordinary and boundary subdivision rules for producing a free-form surface. We do not discuss the latter in this article.)

- **[Wav]** To a number of mathematicians working in wavelet analysis, subdivision is defined as a linear operator $S := S_{\mathbf{a},M}$ of the form:

  $$Sv(\alpha) = \sum_{\beta\in\mathbb{Z}^s} v(\beta)\,\mathbf{a}(\alpha - M\beta), \tag{1.1}$$

  where $\mathbf{a}$ is the mask and M is a so-called isotropic dilation matrix. See [8, 7] and the references therein. This operator has a very close connection to the refinement equation

  $$\phi(x) = \sum_{\beta} \mathbf{a}(\beta)\,\phi(Mx - \beta), \tag{1.2}$$

  which is directly related to wavelet construction under the MRA framework of Mallet and Meyer. Iteratively applying S to a sequence of initial data v produces the data $S^n v$, $n = 0, 1, 2, \ldots$, with

  $$(S^n v)(\alpha) \approx f(M^{-n}\alpha), \quad n \text{ large}, \tag{1.3}$$

  where f is the limit function – exists for a "good" mask $\mathbf{a}$ – of the subdivision process. The geometric entities that underly (1.3) are, of course, the successively refined *lattices*

  $$\mathbb{Z}^s \subset M^{-1}\mathbb{Z}^s \subset M^{-2}\mathbb{Z}^s \subset \cdots \subset \bigcup_{n=0}^{\infty} M^{-n}\mathbb{Z}^s \overset{\text{dense}}{\subset} \mathbb{R}^s. \tag{1.4}$$

Most subdivision schemes we are aware of can be described under both settings. However, it seems unclear whether the two settings are exactly the same in general. In (1.4) only discrete *points* are involved, these points are not connected and hence there is no edge or face, and consequently no concept of tiling/graph/tessellation is directly involved in setting **[Wav]**, so in this sense one may think that **[Wav]** is more general than **[Geo]**. On the other hand, the lattices in (1.4) are all isomorphic to $\mathbb{Z}^s$, whereas the isohedral tiling [T1] involved in **[Geo]** may have no structural similarity whatsoever with $\mathbb{Z}^s$ – so from this point of view **[Geo]** is perhaps more general than **[Wav]**.

At the analysis level, the setting **[Wav]** is very well-understood: there is by now an extensive collection of analytical and computational tools available for the analysis of (1.1)-(1.2).

In this paper, we propose face-based Hermite subdivision schemes which we first describe under the setting **[Geo]**, we then translate them back to the form (1.1) and analyze them using the general theory available. To this end, we reuse the main ideas mutatis mutandis developed in [8]; see Section 2. We follow closely the notations and vocabularies in [8].

We recall here a couple of notations from [8] that will be used very often in the rest of the paper: Λ_r is the set of s-tuples of non-negative integers with sum no greater than r, ordered by the lexicographic ordering. $\mathcal{S}(E, \Lambda_r)$ is the $\#\Lambda_r \times \#\Lambda_r$ matrix that measures how Hermite data of a function changes upon a linear change of variable by $E \in \mathbb{R}^{s\times s}$: let $\partial^{\leq r} f := [D^\nu f]_{\nu\in\Lambda_r} \in \mathbb{R}^{1\times\#\Lambda_r}$, then

$$f \in C^r(\mathbb{R}^s), \;\; g = f(E\cdot) \Longrightarrow \partial^{\leq r} g = \partial^{\leq r} f(E\cdot)\, \mathcal{S}(E, \Lambda_r). \tag{1.5}$$

Why face-based Hermite schemes? Besides theoretical motivations, *some vertex-based schemes are simply unnecessarily smooth for our purposes.* We have been considering applications of symmetric Hermite subdivision schemes in free-form surfaces [13], in which we are primarily interested in schemes that are C^2 and have very small supports. For 2-D Hermite schemes with quincunx refinement, if we insist on using vertex-based scheme then the smallest meaningful support is $[-1, 1]^2$ and the resulted order 1 Hermite scheme is C^7 [8, Section 3.5]; in fact even the scalar counterpart of this scheme is much smoother than the desired C^2: the *scalar* quincunx scheme used in [12] (derived from the Zwart-Powell box spline) has the same $[-1, 1]^2$ support, is a vertex-based scheme and is C^4. If one wants to use the smaller support $[0, 1]^2$ and still obtain schemes with a meaningful symmetry property then a natural approach is to change from vertex- to face-based scheme.

At the initial phase of the research work of this article, we speculated that there existed a C^2 face-based Hermite subdivision scheme with the mask support $[0, 1]^2$ (which corresponds to the stencil in Figure 2(a)) and is based on quincunx refinement, as one could already obtain a C^1 scheme in the scalar case. See Section 3.1. To our surprise, it does not seem to be case. Despite such a "bad news", we shall discover in Section 3.2 a C^2 Hermite face-based subdivision scheme based on quadrisection refinement (i.e. $M = 2I_2$) and the subdivision stencil depicted in Figure 2(a)). The scalar counterpart of this scheme is the regular part of what Peters and Reif called "***the simplest subdivision scheme*** (for smoothing polyhedra)" [11].

From experience it is almost always possible to gain smoothness by increasing support size or by increasing multiplicity – with our experience in Section 3.1 as a rare exception. For application in free-form subdivision surfaces, the latter approach may have an advantage because of the inevitable presence of *extraordinary vertices* in free-form surfaces.

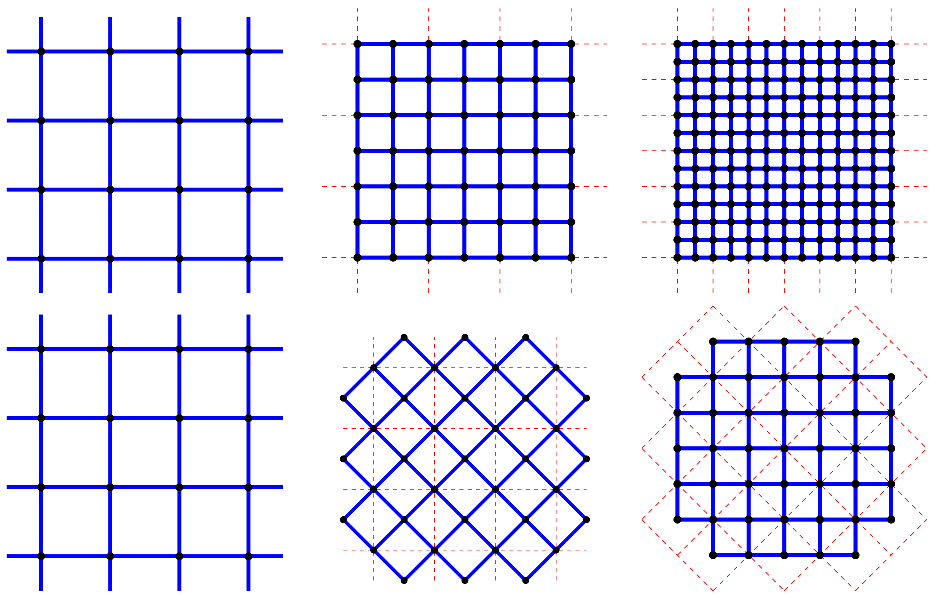

Figure 1: Two topological refinement schemes based on the square tiling

2 Face-based Hermite Subdivision Schemes

In this paper, we consider subdivision schemes based on the following specifications:

[T1] The isohedral tiling of $\mathbb{R}^s$ is chosen to be the straightforward tiling by hypercubes:

$$\mathbb{R}^s = \bigcup_{\alpha \in \mathbb{Z}^s} \alpha + [0,1]^s.$$

[T2] Let M be an $s \times s$ integer matrix such that $M = UDU^T$ where U is unitary and D is diagonal with diagonal entries with the same modulus $\sigma > 1$. In the rest of the paper we simply call any such matrix a *dilation matrix*.[1] Then $M^{-1}[0,1]^s$ is similar to $[0,1]^s$. The successively refined tilings are then given by $\mathbb{R}^s = \bigcup_{\alpha \in \mathbb{Z}^s} M^{-n}(\alpha + [0,1]^s)$. In dimension 2, two well-known examples of M are:

$$2I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_{\text{Quincunx}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

See Figure 1.

[G1] Subdivision data is a "jet" of Hermite data of order r at the **center** of each cube; conceptually, we have

$$\begin{aligned} \mathbb{R}^{1 \times \#\Lambda_r} \ni v_n(\alpha) &= \text{data attached to the tile } M^{-n}([0,1]^s + \alpha) \\ &\approx \partial^{\leq r} f\big(M^{-n}(\alpha + [1/2, \ldots, 1/2]^T)\big)\, \mathcal{S}(M^n, \Lambda_r). \end{aligned} \tag{2.1}$$

See [8] for the exact definition of the notations above; the vector on the right-hand side of (2.1) consists precisely of all the mixed directional derivatives of f of order up to r at the point $M^{-n}(\alpha + [\frac{1}{2}, \ldots, \frac{1}{2}]^T)$ and in directions $M^{-n}e_j$, $j = 1, \ldots, s$.

The informally described concept here will be made precise in definition 2.1 below.

[G2] With the choice of [T1], [T2], [G1] above, there are still many choices of subdivision rules. We are interested in those with small supports, smooth limits and *symmetry*, see the formal definitions below.

In each case above the subdivision scheme can be equivalently defined by a subdivision operator $S = S_{\mathbf{a},M}$ of the form (1.1), i.e. there exists a subdivision operator S such that $v_n := S^n v_0$, $\forall\, n \geq 0$, is precisely what [T1]-[T2] & [G1]-[G2] above would produce. Notice that S is a so-called **stationary** subdivision operator — the subdivision data v_n is generated by a **fixed** subdivision mask $\mathbf{a}$ at all levels n; we mention without a rigorous justification that if we want the data $v_n = S^n v_0$ produced by a stationary subdivision process to also enjoy a natural Hermite type convergence property (c.f. Definition 2.1 below), then the scaling by $\mathcal{S}(M^n, \Lambda_r)$ introduced in (2.1) is essentially the only sensible choice.

In 2-D, here are some subdivision stencils that we are particularly interested in:

1. When $M = M_{\text{Quincunx}}$, the Hermite data associated with any level $n+1$ square is determined from a linear combination of the Hermite data associated with the two level n squares containing the level $n+1$ square. See Figure 2(a). In this case, $\operatorname{supp}(\mathbf{a}) = [0,1] \times [-1,0]$.
2. When $M = 2I_2$, the Hermite data associated with any level $n+1$ square is determined from a linear combination of the Hermite data associated with the 3 or 4 level n squares *closest* to the level $n+1$ square. Figure 2(b)&(c). In this case, $\operatorname{supp}(\mathbf{a}) = [-1,2]^2 - \{(-1,-1), (-1,2), (2,-1), (2,2)\}$ or $[-1,2]^2$, respectively.

Following the geometric intuition we have so far, we are led to the following definition: Let M be a dilation matrix with the property described in [T2] above. An **order r face-based Hermite subdivision operator** $S := S_{\mathbf{a},M}$ is a subdivision operator with multiplicity $m = \#\Lambda_r$ such that (i) for any initial sequence $v \in [l^0(\mathbb{Z}^s)]^{1 \times m}$ there exists $f_v \in C^r(\mathbb{R}^s)$ such that

$$\lim_{n \to \infty} \big\| [\partial^{\leq r} f_v]|_{M^{-n}(\mathbb{Z}^s + [\frac{1}{2}, \ldots, \frac{1}{2}]^T)} - v_n \mathcal{S}(M^n, \Lambda_r) \big\|_{[l^\infty(\mathbb{Z}^s)]^{1 \times m}} = 0, \tag{2.2}$$

[1] Notice that this is more restrictive than what is usually called an isotropic dilation matrix in the literature.

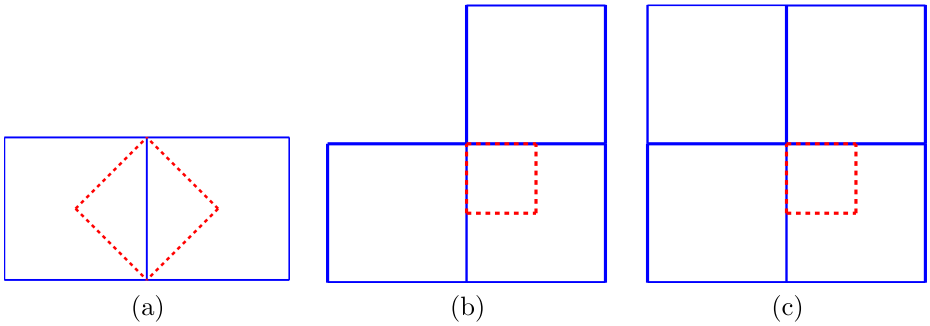

Figure 2: Subdivision Stencils for quincunx (a) and quadrisection (b)&(c) refinement

where $v_n = S^n v$, (ii) $f_v \neq 0$ for some $v \neq 0$. For notational convenience, we write

$$S^\infty v := f_v.$$

We now recall from [8] the following definition, proposed originally for the study of what we now call **vertex-based** Hermite subdivision scheme.

Definition 2.1 ([8, Definition 1.1]) *A subdivision operator $S := S_{\mathbf{a},M}$ is of* **Hermite type** *of order r if (i) $m = \#\Lambda_r$ for some $r \geq 0$, and for any initial sequence $v \in [l^0(\mathbb{Z}^s)]^{1\times m}$ there exists $f_v \in C^r(\mathbb{R}^s)$ such that*

$$\lim_{n\to\infty} ||[\partial^{\leq r} f_v]|_{M^{-n}\mathbb{Z}^s} - v_n \mathcal{S}(M^n, \Lambda_r)||_{[l^\infty(\mathbb{Z}^s)]^{1\times m}} = 0, \tag{2.3}$$

where $v_n = S^n v$, (ii) $f_v \neq 0$ for some $v \neq 0$.

The two definitions are clearly equivalent. Comparing (2.2) and (2.3), of course the only difference is that the latter involves sampling of Hermite data of at the lattices $M^{-n}\mathbb{Z}^s$, whereas the former involves sampling of Hermite data at the *shifted* lattices $M^{-n}(\mathbb{Z}^s + [\frac{1}{2}, \ldots, \frac{1}{2}]^T)$. However, the two convergence definitions involve n tending to infinity, and the difference induced by the shift in (2.2) becomes negligible for large n. An obvious application of triangle inequality, combined with the assumption that $f_v \in C^r$ in both (2.2) and (2.3), shows that the definition of order r face-based Hermite subdivision operator above is identical to Definition 2.1.

So what is the real difference between vertex-based and face-based subdivision schemes then? For the setup in this paper, we address only *sum-rule* and *symmetry* conditions, as these are the two conditions we use for determining a subdivision mask $\mathbf{a}$. We do not address, for instance, data-structure issues for computer implementation.

At first glance,

- Symmetry conditions must be different between vertex- and face-based schemes. As expounded in the introductory section of our earlier paper [8], the meaning of symmetry relies completely on the *geometric meaning* of subdivision data $v_n(\alpha)$; as the "geometric meaning" of $v_n(\alpha)$ has been changed from

  $$v_n(\alpha) \approx \partial^{\leq r} f\big(M^{-n}\alpha\big)\, \mathcal{S}(M^n, \Lambda_r)$$

  in vertex-based scheme to (2.1) in face-based scheme, it should be hardly surprising that symmetry properties for vertex- and face-based Hermite subdivision schemes have to be somewhat different.

  As expected, this is exactly the case. See Section 2.2.

- Sum rule conditions, derived independently of symmetry conditions, are the same for vertex- and face-based Hermite subdivision schemes, because of the equivalence of (2.2) and (2.3).

  As it turns out, this is *not* the case because of a technical loophole in a result in our earlier paper [8]. See Section 2.1.

2.1 Sum rules and the spectral quantity $\nu_\infty(a, M)$

Since a subdivision mask uniquely specifies a subdivision operator, **sum rules** – algebraic relations necessarily satisfied by the mask of any smooth subdivision process – are very useful for *construction* of subdivision schemes.

In the following, let us recall the definitions of sum rules in [2, 5] and the important spectral quantity $\nu_p(\mathbf{a}, M)$ in [5] in the setting of Hermite subdivision schemes.

For a given sequence $u \in (l^0(\mathbb{Z}^s))^{m\times n}$, its *Fourier series* $\hat{u}$ is defined to be

$$\hat{u}(\xi) := \sum_{\beta\in\mathbb{Z}^s} u(\beta)e^{-i\beta\cdot\xi}, \qquad \xi \in \mathbb{R}^s.$$

Let $\mathbf{a}$ be a matrix mask with multiplicity m. We say that $\mathbf{a}$ satisfies the *sum rules* of order $k+1$ with respect to the dilation matrix M (see [5, Page 51]) if there exists a sequence $y \in (l^0(\mathbb{Z}^s))^{1\times m}$ such that $\hat{y}(0) \neq 0$,

$$D^\mu[\hat{y}(M^T\cdot)\hat{\mathbf{a}}(\cdot)](0) = |\det M|D^\mu\hat{y}(0) \quad \text{and} \quad D^\mu[\hat{y}(M^T\cdot)\hat{\mathbf{a}}(\cdot)](2\pi\beta) = 0 \qquad \forall\, |\mu| \leq k, \beta \in (M^T)^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s. \tag{2.4}$$

The quantity $\nu_p(\mathbf{a}, M)$, which is defined in [5, Page 61], plays a very important role in the study of convergence of vector subdivision schemes and the characterization of smoothness of refinable function vectors. For the convenience of the reader, let us recall the definition of the quantity $\nu_p(\mathbf{a}, M)$ from [5, Page 61] as follows. The convolution of two sequences is defined to be

$$[u * v](\alpha) := \sum_{\beta\in\mathbb{Z}^s} u(\beta)v(\alpha-\beta), \qquad u \in (l^0(\mathbb{Z}^s))^{m\times n}, v \in (l^0(\mathbb{Z}^s))^{n\times j}.$$

In terms of the Fourier series, we have $\widehat{u*v} = \hat{u}\hat{v}$. Let y be a sequence in $(l^0(\mathbb{Z}^s))^{1\times m}$. We define the space $\mathcal{V}_{k,y}$ associated with the sequence y by

$$\mathcal{V}_{k,y} := \{v \in (l^0(\mathbb{Z}^s))^{m\times 1} \;:\; D^\mu[\hat{y}(\cdot)\hat{v}(\cdot)](0) = 0 \qquad \forall\, |\mu| \leq k\}. \tag{2.5}$$

Let $1 \leq p \leq \infty$. For any $y \in (l^0(\mathbb{Z}^s))^{1\times m}$ such that $\hat{y}(0) \neq 0$, we define

$$\rho_k(\mathbf{a}, M, p, y) := \sup\left\{\lim_{n\to\infty} \|\mathbf{a}_n * v\|^{1/n}_{(\ell_p(\mathbb{Z}^s))^{m\times 1}} \;:\; v \in \mathcal{V}_{k,y}\right\}, \tag{2.6}$$

where $\mathbf{a}_n$ is defined to be $\hat{\mathbf{a}}_n(\xi) = \hat{\mathbf{a}}((M^T)^{n-1}\xi)\cdots\hat{\mathbf{a}}(M^T\xi)\hat{a}(\xi)$. Define

$$\rho(\mathbf{a}, M, p) := \inf\{\rho_k(\mathbf{a}, M, p, y) \;:\; \text{(2.4) holds for some } k \in \mathbb{N}_0 \text{ and some } y \in (l^0(\mathbb{Z}^s))^{1\times m} \text{ with } \hat{y}(0) \neq 0\}.$$

We define the following quantity:

$$\nu_p(\mathbf{a}, M) := -\log_{\rho(M)}\left[|\det M|^{-1/p}\rho(\mathbf{a}, M, p)\right], \qquad 1 \leq p \leq \infty, \tag{2.7}$$

where $\rho(M)$ denotes the spectral radius of the matrix M. The above quantity $\nu_p(\mathbf{a}, M)$ plays a key role in characterizing the convergence of a vector cascade algorithm in a Sobolev space and in characterizing the L_p smoothness of a refinable function vector. For example, it has showed in [5, Theorem 4.3] that a vector subdivision scheme associated with mask a and dilation matrix M converges in the Sobolev space $W_p^k(\mathbb{R}^s)$ if and only if $\nu_p(\mathbf{a}, M) > k$.

Since the convergence condition (2.2) for face-based Hermite subdivision operator turns out to be *exactly the same* as Definition 2.1 drawn from [8], in deriving sum rule conditions for face-based Hermite subdivision operator, one can presumably simply reuse the result from [8] pertaining to sum rule conditions, which we now recall:

Theorem 2.2 ([8, Theorem 2.2]) *Let M be an isotropic dilation matrix and $\mathbf{a}$ be a mask with multiplicity $m = \#\Lambda_r$. Suppose that $\nu_\infty(\mathbf{a}, M) > r$. Then $S_{\mathbf{a},M}$ is a subdivision operator of Hermite type of order r if $\mathbf{a}$ satisfies the sum rules of order $r+1$ with a sequence $y \in [l^0(\mathbb{Z}^s)]^{1\times\#\Lambda_r}$ such that*

$$\frac{(-iD)^\mu}{\mu!}\hat{y}(0) = e_\mu^T, \qquad \mu \in \Lambda_r. \tag{2.8}$$

At the time article [8] was written, the authors questioned whether the sufficient condition (2.8) is also necessary. If the answer of this open question *were* affirmative, then, in virtue of the equivalence of (2.2) and (2.3), sum rule conditions for vertex- and face-based Hermite subdivision schemes would be exactly the same. However, at the level of generality of Theorem 2.2, **(2.8) is *not* necessary for an order r Hermite type subdivision operator**: Theorem 2.3 below on the one hand generalizes Theorem 2.2 and on the other hand answers the above-mentioned open question *negatively.*

Recall that the cascade operator $Q_{\mathbf{a},M}$ associated with mask $\mathbf{a}$ and dilation matrix M is defined to be

$$Q_{\mathbf{a},M} f := \sum_{\beta \in \mathbb{Z}^s} \mathbf{a}(\beta) f(M \cdot -\beta).$$

A fixed point of this operator is a solution of the refinement equation (1.2). This observation is the starting point of the so-called *strong convergence theory* of refinement equation.

Theorem 2.3 *Let M be an isotropic dilation matrix and* $\mathbf{a}$ *be a mask with multiplicity $m = \#\Lambda_r$. Suppose that $\nu_\infty(\mathbf{a}, M) > r$. Then $S_{\mathbf{a},M}$ is a subdivision operator of Hermite type of order r if* $\mathbf{a}$ *satisfies the sum rules of order $r+1$ with a sequence $y \in [l^0(\mathbb{Z}^s)]^{1\times\#\Lambda_r}$ such that*

$$\frac{(c - iD)^\mu}{\mu!} \hat{y}(0) = e_\mu^T, \qquad \mu \in \Lambda_r \tag{2.9}$$

where $c \in \mathbb{R}^s$ is an arbitrary but fixed vector.

Proof: The proof follows essentially the same line of argument as in the proof of Theorem 2.2, the main new ingredient is the following observation: Let ψ be a Hermite interpolant of order r with accuracy order $r+1$, then

$$\psi_c := \psi(\cdot - c)$$

satisfies moment conditions with respect to a $y \in [l^0(\mathbb{Z}^s)]^{1\times\#\Lambda_r}$ that satisfies (2.9).

By assumption, $\mathbf{a}$ satisfies sum rules of order $r+1$ with a y that also satisfies (2.9), and also that $\nu_\infty(\mathbf{a}, M) > r$, then the strong convergence theory of refinement equation [5, Theorem 4.3] says that

$$\lim_{n\to\infty} ||Q^n_{\mathbf{a},M}\psi_c - \phi||_{[C^r(\mathbb{R}^s)]^{m\times 1}} = 0$$

for some $\phi \in [C^r(\mathbb{R}^s)]^{m\times 1}$; note that ϕ must be a solution of the refinement equation (1.2).

Now we show that $S_{\mathbf{a},M}$ satisfies the Hermite property in Definition 2.1. Let $v \in [l^0(\mathbb{Z}^s)]^{1\times\#\Lambda_r}$, and $v_n = S^n v$. We can also write $v_n = \sum_\beta v(\beta)\mathbf{a}_n(\cdot - M^n\beta)$ where $\mathbf{a}_n = S^n(\delta I_{m\times m})$, $m = \#\Lambda_r$; on the other hand, we have

$$Q^n\psi_c = \sum_\alpha \mathbf{a}_n(\alpha)\psi_c(M^n \cdot -\alpha).$$

Then

$$f_n := \sum_{\alpha\in\mathbb{Z}^s} v_n(\alpha)\psi_c(M^n \cdot -\alpha) = \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} v(\beta)\mathbf{a}_n(\alpha - M^n\beta)\psi_c(M^n \cdot -\alpha) = \sum_{\beta\in\mathbb{Z}^s} v(\beta)(Q^n\psi_c)(\cdot - \beta).$$

Therefore, if $f := \sum_\alpha v(\alpha)\phi(\cdot - \alpha)$, then $\lim_{n\to\infty} ||f_n - f||_{C^r(\mathbb{R}^s)} = 0$ and $\lim_{n\to\infty} ||D^\mu f_n - D^\mu f||_{L^\infty} = 0$, $\forall$ $\mu \in \Lambda_r$.

If we denote by $\partial^{\le r}\psi_c(x)$ the $\#\Lambda_r \times \#\Lambda_r$ matrix with the μ-th row equals to $\partial^{\le r}(\psi_c)_\mu(x)$, then since ψ is a Hermite interpolant,

$$\partial^{\le r}\psi_c(\alpha + c) = I_{\#\Lambda_r\times\#\Lambda_r}\delta(\alpha), \quad \forall\ \alpha \in \mathbb{Z}^s.$$

By (1.5), we have

$$\partial^{\le r} f_n(M^{-n}(\alpha + c)) = \sum_{\beta\in\mathbb{Z}^s} v_n(\beta)(\partial^{\le r}\psi_c)(\alpha + c - \beta)\mathcal{S}(M^n, \Lambda_r) = v_n(\alpha)\mathcal{S}(M^n, \Lambda_r).$$

But then

$$\begin{aligned}
\max_{\alpha\in\mathbb{Z}^s} ||\partial^{\le r} f(M^{-n}(\alpha-c)) - v_n(\alpha)\mathcal{S}(M^n,\Lambda_r)||_\infty &= \max_{\alpha\in\mathbb{Z}^s} ||\partial^{\le r} f(M^{-n}(\alpha-c)) - \partial^{\le r} f_n(M^{-n}(\alpha-c))||_\infty \\
&\le \max_{\mu\in\Lambda_r} ||D^\mu f_n - D^\mu f||_{L^\infty} \to 0, \text{ as } n\to\infty.
\end{aligned}$$

So $\mathbf{a}$ satisfies condition (i) of Definition 2.1 in virtue of the fact that $f \in [C^r(\mathbb{R}^s)]^{m\times 1}$ and

$$\begin{aligned}
||\partial^{\le r} f(M^{-n}\alpha) - v_n(\alpha)\mathcal{S}(M^n,\Lambda_r)||_\infty &\le ||\partial^{\le r} f(M^{-n}\alpha) - \partial^{\le r} f(M^{-n}(\alpha-c))||_\infty + \\
&\quad ||\partial^{\le r} f(M^{-n}(\alpha-c)) - v_n(\alpha)\mathcal{S}(M^n,\Lambda_r)||_\infty \\
&\to 0+0, \text{ as } n\to\infty.
\end{aligned}$$

The condition $\nu_\infty(\mathbf{a}, M) > r$ implies, by [5, Theorem 4.3], that $\mathrm{span}\{\phi(\cdot-\beta) : \beta\in\mathbb{Z}^s\} \supseteq \Pi_r$, which implies $\phi \ne 0$. Thus condition (ii) of Definition 2.1 is also satisfied by $\mathbf{a}$. ■

Theorem 2.3 does not directly tell how to choose the shift vector c. From the proof, however, it seems the most reasonable to choose

$$c = [0,\ldots,0] \quad \text{and} \quad c = [1/2,\ldots,1/2]$$

for vertex and face-based schemes, respectively. We speculate that after one fixes the symmetry condition on mask $\mathbf{a}$ – the topic of the next section – then there corresponds a unique shift vector c such that condition (2.9) is both necessary and sufficient for $S_{\mathbf{a},M}$ being a Hermite type subdivision operator.

2.2 Symmetry

Let G be a finite group of linear maps leaving the cube tiling of $\mathbb{R}^s$ invariant. For a technical reason, we assume also that G is compatible to dilation matrix M in the following sense ([3]):

$$MEM^{-1} \in G, \quad \forall\ E \in G. \tag{2.10}$$

In our bivariate examples, we use exclusively the following symmetry group:

$$D_4 = \left\{ \pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \pm\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \pm\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \pm\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \right\}. \tag{2.11}$$

Note that D_4 is compatible to either $2I_2$ or M_{Quincunx}.

Definition 2.4 *Let G be a symmetry group compatible with dilation matrix M. An order r Hermite subdivision operator has a face-based symmetry property with respect to G if the following condition is satisfied: $\forall\ f \in C^r(\mathbb{R}^s)$ and $E \in G$, if $v := [\partial^{\le r} f]|_{M^{-n}(\mathbb{Z}^s + [\frac{1}{2},\ldots,\frac{1}{2}]^T)}$, $w := [\partial^{\le r} g]|_{M^{-n}(\mathbb{Z}^s + [\frac{1}{2},\ldots,\frac{1}{2}]^T)}$ where $g := f(E\cdot)$, then $S^\infty w = (S^\infty v)(E\cdot)$.*

Theorem 2.5 *An order r Hermite subdivision operator $S_{\mathbf{a},M}$ has a face-based symmetry property with respect to G if*

1. *1 is a simple and dominant eigenvalue of the matrix*

$$J_0 := \sum_{\beta\in\mathbb{Z}^s} a(\beta)/|\det M|$$

and the first entry of its nonzero eigenvector for the eigenvalue 1 is nonzero;

2. *The following symmetry condition on the mask $\mathbf{a}$ holds*

$$\mathbf{a}(E(\alpha - C_{\mathbf{a}}) + C_{\mathbf{a}}) = \mathcal{S}(M^{-1}EM, \Lambda_r)\, \mathbf{a}(\alpha)\, \mathcal{S}(E^{-1}, \Lambda_r), \quad \forall\ E \in G \tag{2.12}$$

where $e := [\frac{1}{2},\ldots,\frac{1}{2}]^T$ and

$$C_{\mathbf{a}} := (M - I_s)\, e. \tag{2.13}$$

Proof: If $\phi = [\phi_1, \dots, \phi_{\#\Lambda_r}]^T$ is the "impulse response" of $S_{\mathbf{a},M}$, then ϕ satisfies the refinement equation (1.2) (that is, $Q_{\mathbf{a},M}\phi = \phi$) and

$$S^\infty v = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)\phi(\cdot - \alpha).$$

We first show that the symmetry condition on $S_{\mathbf{a},M}$ is implied by the following condition on ϕ:

$$\phi(x) = \mathcal{S}(E^{-1}, \Lambda_r)\, \phi(E(x-e)+e). \tag{2.14}$$

(The converse implication is also true and can be easily obtained by adapting part of the proof of [8, Proposition 2.3].)

Assume (2.14). Let f, E, g, v and w be as in Definition 2.4. Then

$$\begin{aligned}
S^\infty w &= \sum_\alpha \partial^{\le r} g(\alpha+e)\,\phi(\cdot-\alpha) \overset{(1.5)}{=} \sum_\alpha \partial^{\le r} f(E(\alpha+e))\,\mathcal{S}(E,\Lambda_r)\,\phi(\cdot-\alpha)\\
&= \sum_\beta \partial^{\le r} f(\beta+e)\,\mathcal{S}(E,\Lambda_r)\,\phi(\cdot-(E^{-1}(\beta+e)-e))\\
&\overset{(2.14)}{=} \sum_\beta \partial^{\le r} f(\beta+e)\,\phi(E\cdot-\beta) = S^\infty v(E\cdot).
\end{aligned}$$

Therefore, the order r Hermite subdivision operator $S_{\mathbf{a},M}$ has a face-based symmetry property with respect to G.

Below we show that (2.14) follows from the assumptions of the theorem; the argument is essentially a time-domain version of the frequency-domain proof of [4, Proposition 2.1].

Let $\phi_E := S(E^{-1},\Lambda_r)\phi(E(\cdot-e)+e), E\in G$. It follows from (2.12) and the definition of $\phi_{MEM^{-1}}$ that

$$\begin{aligned}
Q_{\mathbf{a},M}\phi_{MEM^{-1}} &= \sum_{\alpha\in\mathbb{Z}^s} \mathbf{a}(\alpha)\phi_{MEM^{-1}}(M\cdot-\alpha)\\
&= S(E^{-1},\Lambda_r)\sum_{\alpha\in\mathbb{Z}^s} S(E,\Lambda_r)\mathbf{a}(\alpha)S(ME^{-1}M^{-1},\Lambda_r)\phi(MEM^{-1}(M\cdot-\alpha-e)+e)\\
&= S(E^{-1},\Lambda_r)\sum_{\alpha\in\mathbb{Z}^s} \mathbf{a}(MEM^{-1}(\alpha-C_{\mathbf{a}})+C_{\mathbf{a}})\phi(ME\cdot-MEM^{-1}\alpha-MEM^{-1}e+e)\\
&= S(E^{-1},\Lambda_r)\sum_{\alpha\in\mathbb{Z}^s} \mathbf{a}(\alpha)\phi(ME\cdot-\alpha+C_{\mathbf{a}}-MEM^{-1}C_{\mathbf{a}}-MEM^{-1}e+e)\\
&= S(E^{-1},\Lambda_r)\sum_{\alpha\in\mathbb{Z}^s} \mathbf{a}(\alpha)\phi(M(E\cdot+M^{-1}C_{\mathbf{a}}-EM^{-1}C_{\mathbf{a}}-EM^{-1}e+M^{-1}e)-\alpha)\\
&= S(E^{-1},\Lambda_r)\phi(E\cdot+M^{-1}C_{\mathbf{a}}-EM^{-1}C_{\mathbf{a}}-EM^{-1}e+M^{-1}e).
\end{aligned}$$

By $C_{\mathbf{a}} = (M-I_s)e$, we deduce that

$$\begin{aligned}
M^{-1}C_{\mathbf{a}}-EM^{-1}C_{\mathbf{a}}-EM^{-1}e+M^{-1}e &= (I_s-E)M^{-1}C_{\mathbf{a}}-EM^{-1}e+M^{-1}e\\
&= (I_s-E)M^{-1}(M-I_s)e-EM^{-1}e+M^{-1}e\\
&= (I_s-E)(I_s-M^{-1})e-EM^{-1}e+M^{-1}e\\
&= e-Ee-M^{-1}e+EM^{-1}e-EM^{-1}e+M^{-1}e\\
&= e-Ee.
\end{aligned}$$

We conclude that

$$Q_{\mathbf{a},M}\phi_{MEM^{-1}} = S(E^{-1},\Lambda_r)\phi(E(\cdot-e)+e) = \phi_E \qquad \forall\, E\in G.$$

In other words, we have $Q_{\mathbf{a},M}\phi_E = \phi_{M^{-1}EM}$ for all $E\in G$. Therefore, $Q^n_{\mathbf{a},M}\phi_E = \phi_{M^{-n}EM^n}$ for all $n\in\mathbb{N}$ and $E\in G$.

Since G is a finite group and $M^{-n}EM^n\in G$ for all $n\in\mathbb{N}$, there must exist a positive integer ℓ such that $M^{-\ell}EM^\ell = E$. Consequently, we have $Q^\ell_{a,M}\phi_E = \phi_{M^{-\ell}EM^\ell} = \phi_E$ for some positive integer ℓ. Since 1 is a simple dominant eigenvalue of the matrix J_0, the same can be said to J_0^n for all $n \in \mathbb{N}$. It is known ([10] and references therein) that if J_0^n has 1 as a simple dominant eigenvalue, then ϕ is the unique solution, up to a scalar multiplicative constant, to the refinement equation $Q^n_{a,M}\phi = \phi$.

On the other hand, it follows from the refinement equation that $\hat{\phi}(0)$ and $\widehat{\phi_E}(0)$ are eigenvectors of the matrices J_0 and J_0^ℓ, respectively. Since 1 is a simple eigenvalue of J_0^ℓ, we must have $\widehat{\phi_E}(0) = c\hat{\phi}(0)$ for some complex number $c \in \mathbb{C}$ since $J_0^\ell\widehat{\phi_E}(0) = \widehat{\phi_E}(0)$ and $J_0^n\hat{\phi}(0) = \hat{\phi}(0)$. By the definition of ϕ_E, we have $\widehat{\phi_E}(0) = S(E^{-1}, \Lambda_r)\hat{\phi}(0)$ by $|\det E| = 1$. By our assumption, the first entry in the vector $\hat{\phi}(0)$ is nonzero; that is, $e_1^T\hat{\phi}(0) \neq 0$. Note that the the first row of the matrix $S(E^{-1}, \Lambda_r)$ is e_1^T. We see that $e_1^T\widehat{\phi_E}(0) = e_1^T\hat{\phi}(0) \neq 0$. Therefore, it follows from $\widehat{\phi_E}(0) = c\hat{\phi}(0)$ that c must be 1. Hence, we conclude that we must have $\phi_E = \phi$ by $Q^\ell_{a,M}\phi_E = \phi_E$ and $\widehat{\phi_E}(0) = \hat{\phi}(0)$. In other words, (2.14) holds. ■

3 Examples

In this section, we explore examples in the three cases depicted in Figure 2. In each case, we are interested in bivariate Hermite schemes of order $r = 1$, and we reuse exactly the computational framework developed in [8, Section 3] and the associated MAPLE based solver together with the smoothness optimization code developed in [6]. Underlying this smoothness optimization code is a method by Jia and Jiang [9] which gives the critical L^2 regularity of the refinable function vector ϕ associated with a subdivision mask in the case when ϕ has stable shifts, and a lower bound for the critical L^2 regularity in the absence of stability.

The definition of sum rules, which is given in (2.4) from the frequency domain, can be equivalently rewritten in the time domain as follows: There exists a set of $m \times 1$ vectors $\{Y_\mu \ : \ \mu \in \mathbb{N}_0^s, \ |\mu| \leq k\}$ with $Y_0 \neq 0$ (see [2, Page 22] and [8, Equation (3.9), Section 3]) such that

$$\sum_{0 \leq \nu \leq \mu} (-1)^{|\nu|} Y_{\mu-\nu} J^{\mathbf{a}}_\alpha(\nu) = \sum_{\nu \in \Lambda_k} \mathcal{S}(M^{-1}, \Lambda_k)_{\mu,\nu} Y_\nu \qquad \forall\, \mu \in \Lambda_k, \alpha \in \mathbb{Z}^s,$$

where $(\nu_1, \ldots, \nu_s) \leq (\mu_1, \ldots, \mu_s)$ means $\nu_j \leq \mu_j$ for all $j = 1, \ldots, s$, and $J^{\mathbf{a}}_\alpha(\nu) := \sum_{\beta \in \mathbb{Z}^s} \mathbf{a}(\alpha + M\beta)(\beta + M^{-1}\alpha)^\nu/\nu!$ with $(\xi_1, \ldots, \xi_s)^{(\nu_1, \ldots, \nu_s)} := \xi_1^{\nu_1} \cdots \xi_s^{\nu_s}$. The precise relation between the above definition and the definition of sum rules in (2.4) is

$$Y_\mu = \frac{(-iD)^\mu \hat{y}(0)}{\mu!}.$$

For face-based Hermite subdivision schemes of order $r = 1$ in dimension $s = 2$:

- The sum rules are now with respect to a y that satisfies (2.8) with $c = [\frac{1}{2}, \frac{1}{2}]^T$. For $s = 2$, $r = 1$ and $c = [\frac{1}{2}, \frac{1}{2}]^T$, (2.8) is equivalent to

$$Y_{(0,0)} = [1, 0, 0], \quad Y_{(1,0)} = [1/2, 1, 0], \quad Y_{(0,1)} = [1/2, 0, 1].$$

- Symmetry conditions are those in (2.12)-(2.13).

3.1 $M = M_{\text{Quincunx}}$, $\operatorname{supp}(\mathbf{a}) = [0, 1] \times [-1, 0]$, $G = D_4$

The highest order of sum rule achievable in this case is 3, and the mask we found by our solver has one degree of freedom:

$$\mathbf{a}(0,0) = \begin{bmatrix} \frac{1}{2} & \frac{-4t+1}{2} & \frac{4t-1}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & t & -t \end{bmatrix}$$

and $\mathbf{a}(0, -1)$, $\mathbf{a}(1, -1)$, $\mathbf{a}(1, 0)$ are given by symmetry conditions (2.12)-(2.13). We found, first by empirical observation and then by an explicit calculation, that when $t = -1/4$, the refinable function vector has L^2 smoothness equals to 2.5 and consists of piecewise quadratic C^1 spline functions. By optimizing the L^2

smoothness over the parameter t, we found that when $t = -0.1875$, the L^2 smoothness of the subdivision scheme is 2.57793, only slightly higher than that of the spline scheme.

From a smoothness point of view, one gains almost nothing by going from scalar scheme $r = 0$ to Hermite scheme $r = 1$ – this is atypical when compared to all the examples obtained in [7, 6, 8]. When we choose $r = 0$ with the support size and symmetry group unchanged, the only mask that satisfies the first sum rule is

$$\mathbf{a}(0,0) = \mathbf{a}(1,0) = \mathbf{a}(0,-1) = \mathbf{a}(1,-1) = 1/2, \tag{3.1}$$

its refinable function is a box-spline by Zwart and Powell which is a piecewise quadratic C^1 spline function – same as what the above vector scheme gives when $t = -1/4$. A major difference is that the ZP element has unstable shifts whereas, from various observations, we believe that the refinable function vector of the above spline scheme has stable shifts.

Despite the bad news, we seem to be saved by the good news below.

3.2 $M = 2I_2$, $\operatorname{supp}(\mathbf{a}) \subseteq [-1,2]^2$, $G = D_4$

While (the regular part of) Peters and Reif's *mid-edge scheme* [11] is essentially the quincunx subdivision scheme (3.1), their scheme actually operates based on quadrisection refinement instead of quincunx refinement; this is made possible by the following observation: notice that $M^2_{\text{Quincunx}} = 2I_2$, so if $\mathbf{b} = S^2_{\mathbf{a},M_{\text{Quincunx}}}\delta$, then, in our notations, we have

$$S^2_{\mathbf{a},M_{\text{Quincunx}}} = S_{\mathbf{b},2I_2}.$$

Notice also that $\operatorname{supp}(\mathbf{b}) = [-1,2]^2 - \{(-1,-1),(-1,2),(2,-1),(2,2)\}$. This scheme is C^1 but not C^2. Here, we construct a Hermite version of $S_{\mathbf{b},2I_2}$ which turns out to be C^2.

Using our solver, we found the following 3-parameter mask with the same support as $\mathbf{b}$ above which satisfies sum rules of order 4:

$$\mathbf{a}(0,0) = \begin{bmatrix} 3/4 - 4c_3 & -c_1 + 2c_2 & -c_1 + 2c_2 \\ -3/32 + c_3 & c_1/2 + 5/16 & -c_2 - 1/16 \\ -3/32 + c_3 & -c_2 - 1/16 & c_1/2 + 5/16 \end{bmatrix}, \quad \mathbf{a}(2,0) = \begin{bmatrix} 1/8 + 2c_3 & c_1 & -2c_2 \\ c_3 & 1/8 + c_1/2 & -c_2 \\ -1/32 & 1/16 & 1/16 \end{bmatrix} \tag{3.2}$$

with the other entries given by symmetry conditions. By our smoothness optimization code, the L^2 smoothness of this scheme occurs to be the highest when $(c_1, c_2, c_3) \approx (-7/16, -3/32, 3/64)$, at which the L^2 smoothness is 3.5, thus the Hölder smoothness is at least 2.5, meaning that the scheme is C^2. The associated refinable function vector ϕ is depicted in Figure 3.

support	order of sum rules	# free parameters	highest L^2 Smoothness
12 points	4 (highest possible)	3	3.5
16 points	4	7	3.5
16 points	5 (highest possible)	2	3.0

Table 1: Sum Rules and smoothness attained by some small support face-based Hermite subdivision schemes with $M = 2I_2$

The now-classical Doo-Sabin scheme [1] is based on the tensor product quadratic B-spline, which the latter can be viewed as a face-based scalar subdivision scheme with the 16 point support $[-1,2]^2$. We have explored symmetric face-based Hermite schemes with the same support, which corresponds to the stencil in Figure 2(c). A higher order of sum rules – but not higher smoothness – can be achieved when compared to the 12 point support. See Table 1.

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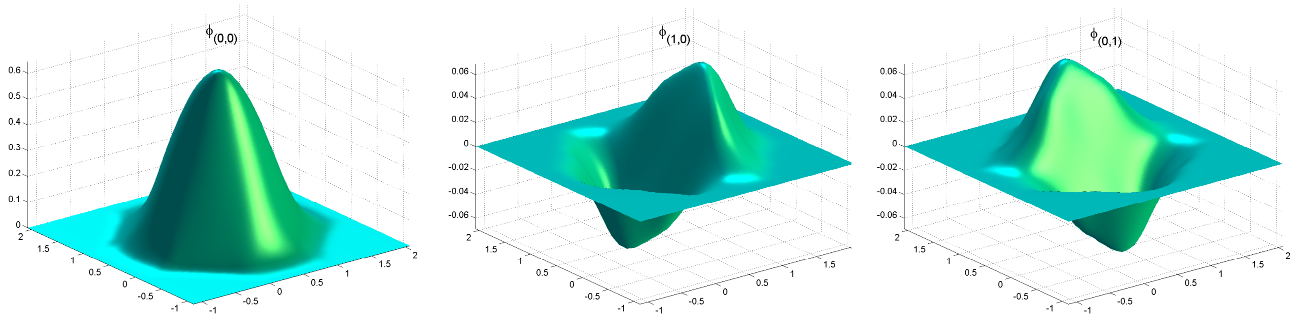

Figure 3: Refinable $\phi = [\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(0,1)}]^T$ associated with the subdivision mask in (3.2)

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