On a Linearization Principle for Nonlinear p-mean Subdivision Schemes

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Abstract:

In a recent preprint [6], the authors consider, motivated by nonlinear signal processing, a family of nonlinear subdivision operators based on interpolation-imputation of p-means. A linearization principle was proposed which, when applied to these nonlinear operators, led to a corresponding family of linear subdivision operators. Based on a number of numerical experiments, the authors conjectured that the critical Hölder regularity of any p-mean subdivision operator is exactly the same as that of the corresponding linear subdivision operator. (The latter can be determined analytically by standard techniques in the linear theory.) In the same article the conjecture was proved in the case of $(p, L, d) = (1, 1, 3)$, i.e. *triadic* subdivision based on *median* interpolation-imputation by *quadratics*. In this note, we prove the conjecture in the case of $(p, L, d) = (\infty, 1, 3)$. The method of proof is by and large similar to that in the $p = 1$ case; in particular the dynamical behaviors of three maps acting on the real projective plane $\mathbb{P}^2$ have to be studied. The main difference is that in the $p = \infty$ case the convergence of an infinite product has to be established during a final stage of the proof.

Keywords. Subdivision Scheme, Nonlinear Subdivision Scheme, Nonlinear Wavelet Transform, Hölder Smoothness, Homogeneous Map, Projective Plane, Median, p-mean, M-Estimator

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1 Introduction

Let $C(I)$ be the Banach space of real-valued continuous functions defined on an interval I, equipped with the uniform norm. The p-mean of a $f \in C(I)$ is defined by

$$m_p(f|I) := \arg\min_{m\in\mathbb{R}} \|f - m\|_{L^p(I)}.$$

Of course, m_2 is the usual average and is a linear functional on $C(I)$; when $p \in [1,\infty]\backslash\{2\}$, m_p is a nonlinear Lipschitz functional on $C(I)$. When $p = 1$, m_1 measures the median of f over I; when $p = \infty$, we have $m_\infty(f|I) = (\max_{x\in I} f(x) + \min_{x\in I} f(x))/2$.

This paper is a companion of the recent preprint [6] in which the authors consider, motivated by construction of robust nonlinear wavelet transforms, a family of nonlinear but affine invariant subdivision operators $S_{p,L,d}$ based on interpolation-imputation of p-means. Precisely, for $p \in [1,\infty]$ and integers $L \geqslant 1$, $d \geqslant 2$, $S_{p,L,d} : l(\mathbb{Z}) \to l(\mathbb{Z})$ is defined as follows:

1. Interpolation: for each $k \in \mathbb{Z}$, let $p_k \in \Pi_{2L}$ be the unique polynomial [5] such that $m_p\left(p_k \mid [k+l, k+l+1]\right) = y_{k+l}$, $l = -L, \dots, L$.
2. Imputation: $\left(S_{p,L,d}(y)\right)_{dk+l} := m_p\left(p_k \mid [k + l/d, k + (l+1)/d\,]\right)$, $0 \leqslant l < d$.

Following [6], d is the *dilation factor* and $2L+1$ is the *locality factor* of $S_{p,L,d}$.

For a nearly straight function $f \in C(I)$ (i.e. $f(x) \approx cx + c'$ for some c, c'), it was shown in [6] using a heuristical argument that $m_p(f|I) \approx \overline{m}_p(f|I)$ where $\overline{m}_p(\cdot|I) : C(I) \to \mathbb{R}$ is the following positive ***linear*** functional

$$\overline{m}_p(f|I = [a,b]) = \int_I f(x)\, d\xi_p(x), \text{ where } \xi_p(x) = \begin{cases} \frac{1}{2}\,\mathrm{sign}(x - \overline{x}), & p = 1\ ; \\ \dfrac{2^{p-2}}{|I|^{p-1}}|x - \overline{x}|^{p-1}\,\mathrm{sign}(x - \overline{x}), & 1 < p < \infty\ ; \\ \frac{1}{2}1_{x=b} - \frac{1}{2}1_{x=a}, & p = \infty. \end{cases} \tag{1.1}$$

In above, $\overline{x}$ is the mid-point of I. Then

$$\overline{m}_1(f|[a,b]) = f((a+b)/2), \quad \overline{m}_\infty(f|[a,b]) = (f(a) + f(b))/2, \quad \text{and}$$

$$\overline{m}_p(f|[a,b]) = \int_a^b f(x) w_p(x)\, dx \quad \text{where } w_p(x) = 2^{p-2}(p-1)|x - (a+b)/2|/(b-a)^{p-1}.$$

Notice that ξ_p, $\overline{m}_p$ and m_p are continuous in $p \in [1,\infty]$.

The *linear* subdivision operator $\overline{S}_{p,L,d}$ is defined exactly the same as $S_{p,L,d}$, but with m_p replaced by $\overline{m}_p$.

It was conjectured in [6] that for at least $L = 1, 2$ and $d = 2, 3$, we have

Conjecture 1.1 *$s_\infty(S_{p,L,d}) = s_\infty(\overline{S}_{p,L,d})$. In particular,*

$$s_\infty(S_{p,1,2}) = s_\infty(\overline{S}_{p,1,2}) = -\log_2 \frac{2+5p}{8(1+p)} \quad \textit{and} \tag{1.2}$$

$$s_\infty(S_{p,1,3}) = s_\infty(\overline{S}_{p,1,3}) = -\log_3 \frac{1+5p}{9(1+p)}, \quad \forall\, p \in [1,\infty]. \tag{1.3}$$

Here $s_\infty(S)$ is the critical Hölder smoothness of the subdivision operator S, see [6, Section 3]. The conjecture is obviously true for $p = 2$ and L and d arbitrary. We have proved in [6] that (1.3) is true when $p = 1$; the main result of this note is that (1.3) is also true when $p = \infty$, i.e.

Proposition 1.2 $s_\infty(S_{\infty,1,3}) = s_\infty(\overline{S}_{\infty,1,3}) = -\log_3 \frac{5}{9}$.

The lower bound $s_\infty(S_{\infty,1,2}) \geqslant -\log_2(2/3)$ is obtained by Oswald [3], the same technique there yields $s_\infty(S_{\infty,1,3}) \geqslant -\log_3(16/27)$ (essentially **[S1] - [S3]** below.)

2 The Proof

In this section, we write $S := S_{\infty,1,3}$, $\overline{S} := \overline{S}_{\infty,1,3}$. Below we follow the notations and terminologies defined in [6]. $S^{[1]}$, $\overline{S}^{[r]}$, $r = 1, 2, 3$, are the unique subdivision operators that satisfy $S^{[1]} \circ \Delta = \Delta \circ S$ and $\overline{S}^{[r]} \circ \Delta^r = \Delta^r \circ \overline{S}$.

1° Under the notation in [6], $S = S_Q$ and $\overline{S} = S_{Q_L}$ where

$$Q_L := \begin{pmatrix} 1/9 & 10/9 & -2/9 \\ -1/9 & 11/9 & -1/9 \\ -2/9 & 10/9 & 1/9 \end{pmatrix} \tag{2.1}$$

and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$Q(m_1, m_2, m_3) := [m_\infty(\pi|[0, 1/3]), m_\infty(\pi|[1/3, 2/3]), m_\infty(\pi|[2/3, 1])]$$

where π is the unique quadratic polynomial that satisfies $m_\infty(\pi|[i-2, i-1]) = m_i$, $i = 1, 2, 3$.

Denote by x^* the extremum point of a $\pi \in \Pi_2$ and $\overline{x}$ the midpoint of $[a, b]$. Using the fact that

$$m_\infty(\pi|[a,b]) = \begin{cases} (f(a) + f(x^*))/2, & \text{if } x^* \in [\overline{x}, b]; \\ (f(x^*) + f(b))/2, & \text{if } x^* \in [a, \overline{x}]; \\ (f(a) + f(b))/2, & \text{otherwise,} \end{cases}$$

one can derive the following closed-form expressions for the map Q:

$$Q(m_1, m_2, m_3) = \begin{cases} m_1 + (m_2 - m_1)\big(q_1(r),\ q_2(r),\ q_3(r)\big) & \text{if } m_1 \neq m_2 \\ \big(\frac{11}{9}m_2 - \frac{2}{9}m_3,\ \frac{10}{9}m_2 - \frac{1}{9}m3,\ \frac{8}{9}m_2 + \frac{1}{9}m_3\big) & \text{if } m_1 = m_2 \end{cases} \tag{2.2}$$

where $r = \frac{m_3 - m_2}{m_2 - m_1}$ and $q_1, q_2, q_3 : \mathbb{R} \to \mathbb{R}$ are defined by:

$$q_1(d), q_2(d) = \begin{cases} \frac{-8d+25+2\sqrt{d^2-d+1}}{27},\ \frac{-d+29-2\sqrt{d^2-d+1}}{27}, & d \in (\frac{8}{3}, \infty); \\ \frac{d^2-12d+12}{36},\ \frac{-d^2+36}{36}, & d \in (2, \frac{8}{3}]; \\ \frac{-2d+8}{9},\ \frac{-d+10}{9}, & d \in (\frac{1}{2}, 2); \\ \frac{12d+1}{18d},\ \frac{36d+1}{36d}, & d \in (\frac{3}{8}, \frac{1}{2}]; \\ \frac{-4d+20+4\sqrt{d^2-d+1}}{27},\ \frac{-2d+28+2\sqrt{d^2-d+1}}{27}, & d \in (0, \frac{3}{8}]; \\ \frac{12d+52+20\sqrt{-3d+1}}{27},\ \frac{24d+68+22\sqrt{-3d+1}}{27}, & d \in (-\frac{13}{25}, 0]; \\ \frac{12d+52+20\sqrt{-3d+1}}{27},\ \frac{63d^2+1074d-1837-(60d+1484)\sqrt{-3d+1}}{324(3d-5-4\sqrt{-3d+1})}, & d \in (-1, -\frac{13}{25}]; \\ \frac{2d+42+20\sqrt{d(d-3)}}{81},\ \frac{-217d^2+1722d-909+(188d-1236)\sqrt{d(d-3)}}{324(5d-3-4\sqrt{d(d-3)})} & d \in (-\frac{25}{13}, -1]; \\ \frac{-d+3}{4},\ \frac{13d+57+22\sqrt{d(d-3)}}{81} & d \in (-\frac{121}{25}, -\frac{25}{13}]; \\ -\frac{88d^2-417d+207-(74d-282)\sqrt{d(d-3)}}{81(5d-3-4\sqrt{d(d-3)})},\ \frac{13d+57+22\sqrt{d(d-3)}}{81} & d \in (-\infty, -\frac{121}{25}]; \end{cases} \tag{2.3}$$

and $q_3(d) = 1 + d - dq_1(1/d)$ if $d \neq 0$ and $q_3(0) = \lim_{d \to 0} q_3(d) = \frac{11}{9}$.

2° For $m \in \mathbb{R}^3$,

$$Q(m) = Q_L(m) \quad \text{if} \quad m_1 = m_2 \quad \text{or} \quad m_2 = m_3 \quad \text{or} \quad \frac{m_3 - m_2}{m_2 - m_1} \in [\frac{1}{2}, 2]. \tag{2.4}$$

We claim that

$$\exists\, C > 0 \quad \text{s.t.} \quad \|(Q - Q_L)m\|_\infty \leqslant C|\Delta^2 m|, \quad \forall\, m \in \mathbb{R}^3. \tag{2.5}$$

This is a direct consequence of the facts that Q, Q_L are both (i) continuous, (ii) offset invariant, (iii) homogeneous and (iv) invariant under $P_r := \{m \in \mathbb{R}^3 : \Delta^r m = 0\}$, $r = 1, 2$. Recall from [6] the notation $\mathbb{P}(\mathbb{R}^2) = \mathbb{P}^1 \cup \{[0, 0]_\sim\}$. Consider $R : \mathbb{P}^1 \to \mathbb{R}$ defined by

$$R([\Delta m]_\sim) = \begin{cases} \dfrac{\|(Q - Q_L)m\|_\infty}{|\Delta^2 m|}, & \text{if } m \notin P_2, \\ 0, & \text{if } m \in P_2; \end{cases}$$

one sees from (2.4) and (i)-(iv) above that R is a *well-defined continuous function on the compact space* $\mathbb{P}^1$, and hence has a maximum value C. This establishes (2.5).

By the general results mentioned in [6, Section 3], (2.5) implies that the Hölder regularity of S can be determined from the decay rate of $\|\Delta^2 S^j v\|_{l^\infty}$; in particular, if we can establish the bound

$$\exists\, C>0 \quad \text{s.t.} \quad \|\Delta^2 S^j v\|_{l^\infty} \leqslant C\|v\|_{l^\infty}(5/9)^j, \quad \forall\, v\in l^\infty \tag{2.6}$$

then

$$s_\infty(S) \geqslant -\log_3(5/9) = s_\infty(\overline{S}) \tag{2.7}$$

follows.

3° One way to prove $s_\infty(S)\leqslant -\log_3(5/9)$ is as follows. By a straightforward adaption of the argument in the proof of the converse part of [1, Theorem 3.4], one can show that if $s_\infty(S) > -\log_3(5/9)$, then

$$\|\Delta S^j m\|_{l^\infty} = o((5/9)^j) \quad \forall\, m\in l^\infty. \tag{2.8}$$

Using the closed-form formulas (2.3), one can calculate that if $m=(\cdots,0,0,1,1,\cdots)$, then the left-hand side of (2.8) does not decay faster than $(5/9)^j$; this calculation can be facilitated by the existence of $S^{[1]}$ and the associated closed-form expressions, see below.

4° S is an **order 2** affine invariant subdivision operator with **neighborhood factor** $R_S=4$; also we have $R_{S^{[1]}}=3$. Let $m\in\mathbb{R}^4$, then [1]

$$S(m)=$$
$$\begin{cases} (\frac{1}{9}m_1+\frac{8}{9}m_2,\ \frac{10}{9}m_2-\frac{1}{9}m_1,\ \frac{11}{9}m_2-\frac{2}{9}m_1,\ \frac{11}{9}m_2-\frac{2}{9}m_4,\ \frac{10}{9}m_2-\frac{1}{9}m_4,\ \frac{8}{9}m_2+\frac{1}{9}m_4) & \text{if } m_2=m_3\\ (m_3-(m_3-m_2)(q_3(x),\ q_2(x),q_1(x)),\ m_2+(m_3-m_2)(q_1(y),\ q_2(y),q_3(y))) & \text{otherwise}\end{cases} \tag{2.9}$$

where $x=\frac{m_2-m_1}{m_3-m_2}$ and $y=\frac{m_4-m_3}{m_3-m_2}$. Since $S^{[1]}\circ\Delta=\Delta\circ S$, by substituting $d_i=m_{i+1}-m_i$, $i=1,2,3$, into (2.9), we get

$$S^{[1]}(d_1,d_2,d_3)=$$
$$\begin{cases} \frac{1}{9}\big(2d_1,\ d_1,\ -2(d_1+d_3),\ d_3,\ 2d_3\big), & \text{if } d_2=0;\\ d_2\big(q_3(x)-q_2(x),\ q_2(x)-q_1(x),\ q_1(x)+q_1(y)-1,\ q_2(y)-q_1(y),\ q_3(y)-q_2(y)\big), & \text{otherwise}\end{cases} \tag{2.10}$$

where $x=d_1/d_2$ and $y=d_3/d_2$.

Being a triadic homogeneous subdivision operator with neighborhood factor 3, $S^{[1]}$ can be represented by three homogeneous maps $P^i:\mathbb{R}^3\to\mathbb{R}^3$, $i=1,2,3$, see [6, Proposition 2.2]; we will be working with the quotient maps $\varrho_i := [P^i]:\mathbb{P}(\mathbb{R}^3)\to\mathbb{P}(\mathbb{R}^3)$, $i=1,2,3$. Since each element in $\mathbb{P}(\mathbb{R}^3)$ can be represented as either $[(x,0,y)]_\sim$ or $[(x,1,y)]_\sim$, by (2.10) ϱ_i have the following closed-form expressions:

$$\varrho_1(\theta)=\begin{cases} [(2x,x,-2x-2y)]_\sim, & \text{if } \theta=[(x,0,y)]_\sim;\\ [(q_3(x)-q_2(x),q_2(x)-q_1(x),q_1(x)+q_1(y)-1)]_\sim, & \text{if } \theta=[(x,1,y)]_\sim.\end{cases} \tag{2.11}$$

$$\varrho_2(\theta)=\begin{cases} [(x,-2x-2y,y)]_\sim, & \text{if } \theta=[(x,0,y)]_\sim;\\ [(q_2(x)-q_1(x),q_1(x)+q_1(y)-1,q_2(y)-q_1(y))]_\sim, & \text{if } \theta=[(x,1,y)]_\sim.\end{cases} \tag{2.12}$$

$$\varrho_3(\theta)=\begin{cases} [(-2x-2y,y,2y)]_\sim, & \text{if } \theta=[(x,0,y)]_\sim;\\ [(q_1(x)+q_1(y)-1,q_2(y)-q_1(y),q_3(y)-q_2(y))]_\sim, & \text{if } \theta=[(x,1,y)]_\sim.\end{cases} \tag{2.13}$$

Define the maps of "shrinking factors" $\varsigma_l:\mathbb{P}(\mathbb{R}^3)\to[0,\infty)$

$$\varsigma_l([d]_\sim) := \begin{cases} \dfrac{\|\Delta P^l(d)\|_\infty}{\|\Delta d\|_\infty}, & \text{if } \Delta d\neq 0;\\ 0, & \text{otherwise;}\end{cases}$$

[1] If x is a finite length vector, $S(x)$ is "all the data that S can generate by knowing *only* the segment x of a sequence" [6, Section 2].

observe that the homogenity of P^l implies that ς_l is well-defined. Let $\varsigma : \mathbb{P}(\mathbb{R}^3) \to [0,\infty)$ be defined by $\varsigma(\theta) := \max_{l=1,2,3} \varsigma_l(\theta)$. Then by (2.10),

$$\varsigma(\theta) = \begin{cases} \dfrac{1}{9}\dfrac{\max(|x|,\ |3x+2y|,\ |2x+3y|,\ |y|)}{\max(|x|,\ |y|)}, & \text{if } \theta = [x,0,y]_\sim \text{ and } (x,y) \neq (0,0); \\ \dfrac{\max(D(x,y), D(y,x))}{\max(|x-1|,\ |y-1|)}, & \text{if } \theta = [x,1,y]_\sim \text{ and } (x,y) \neq (1,1); \\ 0, & \text{otherwise}; \end{cases} \tag{2.14}$$

where

$$D(x,y) := \max\left(|q_1(x) - 2q_2(x) + q_3(x)|, |2q_1(x) + q_1(y) - q_2(x) - 1|\right). \tag{2.15}$$

One can verify that $\varsigma([x,0,y]_\sim) = \lim_{c\to\infty} \varsigma([cx,1,cy]_\sim)$ when $(x,y) \neq (0,0)$. We also abuse notation and write

$$\varsigma(x,y) := \varsigma([x,1,y]_\sim).$$

By [6, Theorem 3.4], to prove (2.6) it suffices to show that there exists an absolute constant $C > 0$ such that for any $\theta \in \mathbb{P}(\mathbb{R}^3)$ and any sequence $(\varepsilon_i)_{i=1}^\infty$, $\varepsilon_i \in \{1,2,3\}$, if we define

$$\theta_0 := \theta, \quad \theta_j := \varrho_{\varepsilon_j}(\theta_{j-1}), \tag{2.16}$$

then for any $j \geqslant 1$,

$$\prod_{k=0}^{j-1} \varsigma(\theta_k) \leqslant C\ (5/9)^{-j}. \tag{2.17}$$

5° *If* $\varsigma(\theta) \leqslant 5/9$ for all θ, then (2.17) holds for a trivial reason; unfortunately the "bad zone"

$$\mathcal{B} := \{\theta \in \mathbb{P}(\mathbb{R}^3) : \varsigma(\theta) > 5/9\}$$

is nonempty. (See Figure 1.)

Consider the partition $\mathbb{P}(\mathbb{R}^3) = B \cup G$ where

$$B := \{[(a_1,a_2,a_3)]_\sim \in \mathbb{P}(\mathbb{R}^3) : (a_1 - a_2)(a_3 - a_2) > 0\} \quad \text{and}$$

$$G := \{[(a_1,a_2,a_3)]_\sim \in \mathbb{P}(\mathbb{R}^3) : (a_1 - a_2)(a_3 - a_2) \leqslant 0\}.$$

Let

$$B_1 := \{[(x,1,y)]_\sim : (x,y) \in (-\infty,-4]^2\}, \quad B_2 := \{[(x,1,y)]_\sim : (x,y) \in [0,\tfrac{1}{2}]^2\},$$

$$B_{21} := \{[(x,1,y)]_\sim : (x,y) \in [\tfrac{3}{8},\tfrac{1}{2}]^2\}.$$

In the next two subsections we shall prove the following facts:

[D1] (i) $\varrho_i(B) \subset G$ for $i = 1,3$, (ii) $\varrho_2(B) \subset B$.

[D2] For any $\theta \in \mathbb{P}(\mathbb{R}^3)$, $\mathrm{orbit}(\theta,\varrho_2) := \{\varrho_2^n(\theta) : n = 0,1,\ldots\}$ satisfies

$$|\mathrm{orbit}(\theta,\varrho_2) \cap B_1| \leqslant 1, \quad |\mathrm{orbit}(\theta,\varrho_2) \cap (B_2 \backslash B_{21})| \leqslant 2.$$

[S1] $\mathcal{B} \subset B_1 \cup B_2 \subset B$.

[S2] $\max_{\theta\in B_1} \varsigma(\theta) = \varsigma(-121/25,-121/25) = 41/73$ $(> 5/9.)$

[S3] $\max_{\theta\in B_2} \varsigma(\theta) = \varsigma(3/8,3/8) = 16/27$ $(> 5/9.)$

[S4] $\max_{\theta\in G} \varsigma(\theta) = \varsigma(3/8,1) = \varsigma(1,3/8) = 16/45$ $(< 5/9.)$

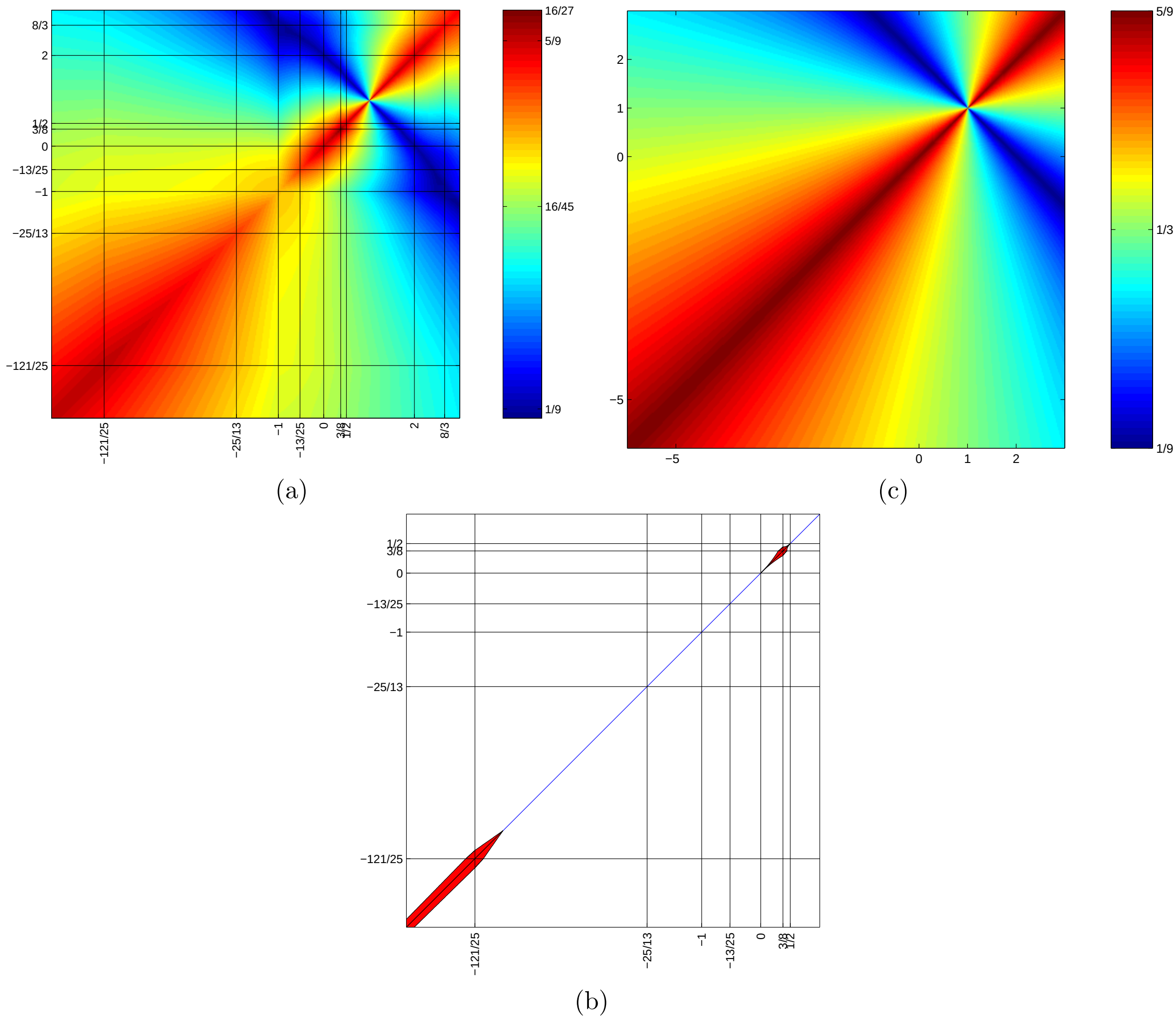

Figure 1: (a) Shrinking factor $\varsigma(x,y)$, (b) Bad zone $\mathcal{B}=\{(x,y):\varsigma(x,y)>\frac{5}{9}\}\subset B_1\cup B_2$. These figures are produced by using (2.14) and (2.3), based on dividing the (x,y) plane into 10×10 rectangles $I\times J$, where I and J are one of $(-\infty,-121/25]$, $[-121/25,-25/13]$, $[-25/13,-1]$, $[-1,-13/25]$, $[-13/25,0]$, $[0,3/8]$, $[3/8,1/2]$, $[1/2,2]$, $[2,8/3]$, $[8/3,\infty)$. (c) "Linear Shrinking Factor" $\varsigma_{_L}(x,y)$, see (2.20).

[S5] For any $\theta\in B_{21}$, for any integer $N\geqslant 0$,

$$\prod_{n=0}^{N}\varsigma(\varrho_2^n(\theta))\leqslant 1.35\times(\frac{5}{9})^{N+1}. \tag{2.18}$$

The above facts combine to yield (2.17) with $C=C^*:=41/73\times(16/27)^2\times(9/5)^3\times 1.35$. This can be seen as an adaption of the "compensation argument" in [6, Section 4] to the present situation. The key point is that the "good factor" in **[S4]** is good enough to "compensate" all the "bad factors" in **[S2]**, **[S3]** and **[S5]**: $41/73\times(16/27)^2\times(9/5)^3\times 1.35\times 16/45\times 9/5<1$.

Comment. One may notice that the proof here is essentially identical to that of the $p=1$ case found in [6, Section 4], the key difference is that in the $p=\infty$ case we have the following "unfortunate" facts: for *almost all* $\theta\in\mathbb{P}(\mathbb{R}^3)$,

$$|\{\mathrm{orbit}(\theta,\varrho_{_2})\cap\mathcal{B}\}|=\infty,\quad \lim_{j\to\infty}\varrho_{_2}^j(\theta)=[1/2,1,1/2]_\sim,\quad [1/2,1,1/2]_\sim\in\partial\mathcal{B}.$$

These necessitate the careful analysis of the *infinite product* involved in **[S5]**. In the $p=1$ case, we have instead $\lim_{j\to\infty}\varrho_{_2}^j(\theta)=[1,1,1]_\sim$, this limit point is well at the exterior of the corresponding bad zone $\mathcal{B}$, and $|\{\text{orbit}(\theta, \varrho_2) \cap \mathcal{B}\}| \leqslant 2$.

2.1 Dynamics of ϱ_i

From (2.3), one can show by elementary calculus that

Lemma 2.1 *q_1, $q_1 - q_2$ and $-q_1 + 2q_2 - q_3$ are strictly decreasing.*

Proof of [D1]: For part (i), by symmetry it suffices to prove the proposition for $i = 1$: if we define $\mathcal{R} : \mathbb{P}(\mathbb{R}^3) \to \mathbb{P}(\mathbb{R}^3)$ by $\mathcal{R}([d_1, d_2, d_3]_\sim) := [d_3, d_2, d_1]_\sim$, then $\mathcal{R}(G) = G$, $\mathcal{R}(B) = B$ and $\varrho_3 \circ \mathcal{R} = \mathcal{R} \circ \varrho_1$; thus if the proposition holds for $i = 1$, then

$$\varrho_3(B) = \varrho_3(\mathcal{R}(B)) = \mathcal{R}(\varrho_1(B)) \subset \mathcal{R}(G) = G.$$

Let $\theta = [(a_1, a_2, a_3)]_\sim \in B$. If $a_2 = 0$, then $a_1 a_3 > 0$ and $\varrho_1(\theta) = [(2a_1, a_1, -2a_1 - 2a_3)]_\sim$. Since $(2a_1 - a_1)(-2a_1 - 2a_3 - a_1) = -3a_1^2 - 2a_1 a_3 < 0$, $[(2a_1, a_1, -2a_1 - 2a_3)]_\sim \in G$.

If $a_2 \neq 0$, then $[(a_1, a_2, a_3)]_\sim = [(x, 1, y)]_\sim$, where $x = \frac{a_1}{a_2}, y = \frac{a_3}{a_2}$. Since $[(x, 1, y)]_\sim \in B$, we have $(x - 1)(y - 1) > 0$. Since $\varrho_1([(x, 1, y)]_\sim) = [(q_3(x) - q_2(x), q_2(x) - q_1(x), q_1(x) + q_1(y) - 1)]_\sim$, in order to prove $\varrho_1([(x, 1, y)]_\sim) \in G$, we need to show $(q_1(x) + q_3(x) - 2q_2(x))(2q_1(x) - q_2(x) + q_1(y) - 1) \leqslant 0$.

By monotonicity, when $x > 1$ and $y > 1$, $q_1(x) + q_3(x) - 2q_2(x) > q_1(1) + q_3(1) - 2q_2(1) = 0$ and $2q_1(x) - q_2(x) + q_1(y) - 1 < 2q_1(1) - q_2(1) + q_1(1) - 1 = 0$; when $x < 1$ and $y < 1$, $q_1(x) + q_3(x) - 2q_2(x) < q_1(1) + q_3(1) - 2q_2(1) = 0$ and $2q_1(x) - q_2(x) + q_1(y) - 1 > 2q_1(1) - q_2(1) + q_1(1) - 1 = 0$. Consequently, $(q_1(x) + q_3(x) - 2q_2(x))(2q_1(x) - q_2(x) + q_1(y) - 1) \leqslant 0$ when $(x - 1)(y - 1) > 0$, so $\varrho_1([(x, 1, y)]_\sim) \in G$.

Part **[D1]**(ii) can be proved similarly. ∎

Lemma 2.2 *ϱ_2 maps B_2 and B_{21} into itself.*

Proof: By (2.12),

$$\varrho_2([(x, 1, y)]_\sim) = [(q_2(x) - q_1(x),\ q_1(x) + q_1(y) - 1,\ q_2(y) - q_1(y))]_\sim. \tag{2.19}$$

If $\theta \in B_3$, then $\theta = [x, 1, y]_\sim$ with $x, y \in [0, \frac{1}{2}]$, consequently monotonicity gives $q_2(x) - q_1(x), q_2(y) - q_1(y) \in [q_2(0) - q_1(0), q_2(\frac{1}{2}) - q_1(\frac{1}{2})] = [\frac{2}{9}, \frac{5}{18}]$, and $q_1(x) + q_1(y) - 1 \in [2q_1(\frac{1}{2}) - 1, 2q_1(0) - 1] = [\frac{5}{9}, \frac{7}{9}]$. Therefore $\frac{q_2(x)-q_1(x)}{q_1(x)+q_1(y)-1}$, $\frac{q_2(y)-q_1(y)}{q_1(x)+q_1(y)-1} \in [\frac{2}{7}, \frac{1}{2}]$ and $\varrho_2(\theta) \in B_2$. Thus $\varrho_2(B_2) \subset B_2$.

Similar calculations show that $\varrho_2(B_{21}) \subset B_{21}$. ∎

Proof of [D2]: We first prove $|\text{orbit}(\theta, \varrho_2) \cap B_1| \leqslant 1$. Assume $\theta = [(x, 1, y)]_\sim \in B_1$, we are going to prove $\{\varrho_2^n(\theta) : n = 1, 2, \cdots\} \cap B_1 = \emptyset$. More specifically, we will prove

(1) $\varrho_2(\theta) \in \{[(x, 1, y)]_\sim : (x, y) \in [-\frac{1}{2}, 0]^2\}$;

(2) $\{\varrho_2^n(\theta) : n = 2, 3, \cdots\} \subset B_2$.

Suppose $\varrho_2(\theta) = [(x_1, 1, y_1)]_\sim$. We are going to prove $x_1 \in [-\frac{1}{2}, 0]$. $y_1 \in [-\frac{1}{2}, 0]$ can be obtained similarly. From (2.12), we have $x_1 = \frac{q_2(x)-q_1(x)}{q_1(x)+q_1(y)-1}$. Since $\theta = [(x, 1, y)]_\sim \in B_1$, we have $x \leqslant -4, y \leqslant -4$. By Lemma 2.1, we can get $q_2(x) - q_1(x) \leqslant q_2(-4) - q_1(-4) < 0$ and $q_1(x) + q_1(y) - 1 \geqslant 2q_1(-4) - 1 > 0$. So $\frac{q_2(x)-q_1(x)}{q_1(x)+q_1(y)-1} < 0$. To prove $\frac{q_2(x)-q_1(x)}{q_1(x)+q_1(y)-1} > -\frac{1}{2}$, we only need to prove $2(q_2(x) - q_1(x)) \geqslant -(q_1(x) + q_1(y) - 1)$, i.e. $2q_2(x) - q_1(x) + q_1(y) \geqslant 1$, which can be verified by (2.3).

By Lemma 2.2, to prove (2), it suffices to prove $\varrho_2^2(\theta) \in B_2$. Suppose $\varrho_2^2(\theta) = [(x_2, 1, y_2)]_\sim$, then $x_2 = \frac{q_2(x_1)-q_1(x_1)}{q_1(x_1)+q_1(y_1)-1}$. Since $x_1, y_1 \in [-\frac{1}{2}, 0]$, it follows from Lemma 2.1 that

$$0 < \frac{q_2(-\frac{1}{2}) - q_1(-\frac{1}{2})}{q_1(-\frac{1}{2}) + q_1(-\frac{1}{2}) - 1} \leqslant \frac{q_2(x_1) - q_1(x_1)}{q_1(x_1) + q_1(y_1) - 1} \leqslant \frac{q_2(0) - q_1(0)}{q_1(0) + q_1(0) - 1} = \frac{2}{7}.$$

Thus $x_2 \in [0, \frac{1}{2}]$. Similarly, we can prove $y_2 \in [0, \frac{1}{2}]$. Therefore $\varrho_2^2(\theta) \in B_2$.

To prove $|\mathrm{orbit}(\theta, \varrho_2) \cap (B_2 \backslash B_{21})|$, it suffices to prove $\varrho_2^2(\theta) \in B_{21}$ for any $\theta \in B_2$.

Suppose $\theta = [(x, 1, y)]_\sim \in B_2$, $\varrho_2(\theta) = [(x_1, 1, y_1)]_\sim$ and $\varrho_2^2(\theta) = [(x_2, 1, y_2)]_\sim$. Since $x, y \in [0, \frac{1}{2}]$, it follows from Lemma 2.1 that

$$\frac{2}{7} = \frac{q_2(0) - q_1(0)}{q_1(0) + q_1(0) - 1} \leqslant \frac{q_2(x) - q_1(x)}{q_1(x) + q_1(y) - 1} \leqslant \frac{q_2(\frac{1}{2}) - q_1(\frac{1}{2})}{q_1(\frac{1}{2}) + q_1(\frac{1}{2}) - 1} = \frac{1}{2}.$$

So $x_1 \in [\frac{2}{7}, \frac{1}{2}]$. Similarly, we can prove $y_1 \in [\frac{2}{7}, \frac{1}{2}]$. Thus

$$\frac{3}{8} < \frac{q_2(\frac{2}{7}) - q_1(\frac{2}{7})}{q_1(\frac{2}{7}) + q_1(\frac{2}{7}) - 1} \leqslant \frac{q_2(x_1) - q_1(x_1)}{q_1(x_1) + q_1(y_1) - 1} \leqslant \frac{q_2(\frac{1}{2}) - q_1(\frac{1}{2})}{q_1(\frac{1}{2}) + q_1(\frac{1}{2}) - 1} = \frac{1}{2}.$$

Therefore $x_2 \in [\frac{3}{8}, \frac{1}{2}]$. Similarly we can obtain $y_2 \in [\frac{3}{8}, \frac{1}{2}]$. Consequently, $\varrho_2^2(\theta) \in B_{21}$. ■

2.2 Distribution of ς

One can verify **[S1]-[S4]** in a brute-force manner: the formulas (2.3) suggest to divide the (x, y) plane into 100 rectangles $I \times J$ (see Figure 1) so that $\max_{(x,y) \in I \times J} \varsigma(x, y)$ can be determined using bivariate calculus after verifying certain inequalities. In order to avoid checking the hundreds of inequalities needed in this brute-force approach, we observe that S_Q and S_{Q_L} share certain qualitative similarities, despite their quantitative difference.

If $\varsigma_L : \mathbb{P}(\mathbb{R}^3) \to [0, \infty)$ is defined similarly as ς but with $S_Q^{[1]}$ replaced by the linear $S_{Q_L}^{[1]}$, then if $(x, y) \neq (1, 1)$,

$$\varsigma_L([x, 1, y]_\sim) = \frac{\max(D_L(x, y), D_L(y, x))}{\max(|x - 1|, |y - 1|)} \tag{2.20}$$

where $D_L(x, y) = \frac{1}{9} \max(|x - 1|, |3(x - 1) + 2(y - 1)|)$.

One can see either directly from the above expression or, more fundamentally, from the existence of $\overline{S}^{[2]}$ that if $s \neq 0$ and $|c| \leqslant 1$ then

$$\varsigma_L([s + 1, 1, cs + 1]_\sim) = \frac{3 + 2c}{9}.$$

In particular, ς_L is (i) constant in s and (ii) increases from 1/9 to 5/9 as c increases from -1 to 1. For ς, nonlinearity present in S_Q spoils these properties, but we still have:

Lemma 2.3 *For any $s \in \mathbb{R}$,*

(i) $\varsigma([s + 1, 1, cs + 1]_\sim)$ *is increasing in c when $c \in [0, 1]$,*

(ii) $\varsigma([s + 1, 1, cs + 1]_\sim) \leqslant \varsigma([s + 1, 1, 1]_\sim)$ *for $c \in [-1, 0]$.*

The proof is almost identical to that of [6, Lemma 4.4] and we omit it.

It follows from this Lemma and $\varsigma(x, y) = \varsigma(y, x)$ that

$$\varsigma(x, y) \leqslant \max(\varsigma(x, x), \varsigma(y, y)) \tag{2.21}$$

for all $(x, y) \in \mathbb{R}^2$.

Proof of [S4]: Let $G_0 := \{[(x, 0, y)]_\sim : xy \leqslant 0\}$ and $G_1 := \{[(x, 1, y)]_\sim : (x - 1)(y - 1) \leqslant 0\}$, then

$$\begin{aligned} G &= \{[(a_1, a_2, a_3)]_\sim \in \mathbb{P}(\mathbb{R}^3) : (a_1 - a_2)(a_3 - a_2) \leqslant 0\} \\ &= G_0 \cup G_1. \end{aligned}$$

When $\theta = [(x,0,y)]_\sim \in G_0$, we have $xy \leqslant 0$. So $|3x+2y| \leqslant \max(3|x|, 2|y|)$ and $|2x+3y| \leqslant \max(2|x|, 3|y|)$. It follows that $\max(|x|, |3x+2y|, |2x+3y|, |y|) \leqslant 3\max(|x|, |y|)$. So, by (2.14), $\varsigma(\theta) \leqslant \frac{1}{3}$. When $\theta = [(x,1,y)]_\sim \in G_1$, we can assume $(x,y) \neq (1,1)$ as $\varsigma(1,1) = 0$. Notice also the symmetry property $\varsigma(x,y) = \varsigma(y,x)$. So we have

$$\max_{\theta \in G_1} \varsigma(\theta) = \max_{\substack{(x-1)(y-1)\leqslant 0\\(x,y)\neq(1,1)}} \varsigma(x,y) = \max_{\substack{-1\leqslant \frac{y-1}{x-1}\leqslant 0\\ x\neq 1}} \varsigma(x,y) = \max_{\substack{c\in[-1,0]\\ s\in\mathbb{R}\\ s\neq 0}} \varsigma(s+1, cs+1) \overset{\text{Lemma2.3(ii)}}{=} \max_{\substack{s\in\mathbb{R}\\ s\neq 0}} \varsigma(s+1,1).$$

From (2.14) and (2.15), for $s \neq 0$

$$\begin{aligned}\varsigma(s+1,1) &= \frac{\max(D(s+1,1), D(1,s+1))}{|s|}\\ &= \frac{\max(|q_1(s+1) - 2q_2(s+1) + q_3(s+1)|, |2q_1(s+1) - q_2(s+1) - \frac{1}{3}|, |q_1(s+1) - \frac{2}{3}|)}{|s|}.\end{aligned}$$

With computational details omitted, one can show using (2.3) that the 1-D maximization problem $\sup_{s\neq 0} \varsigma(s+1,1)$ has the solution 16/45 and the unique maximizer $s = -5/8$. Summarizing,

$$\max_{\theta\in G} \varsigma(\theta) = \max(\max_{\theta\in G_0} \varsigma(\theta), \max_{\theta\in G_1} \varsigma(\theta)) = \max(1/3, 16/45) = 16/45 = \varsigma(3/8, 1) = \varsigma(1, 3/8).$$

■

Proof of [S1]: If $\theta = [(x,0,y)]_\sim$, then, by (2.14), $\varsigma(\theta) \leqslant \frac{5}{9}$.

So $\mathcal{B} = \{\theta \in \mathbb{P}(\mathbb{R}^3) : \varsigma(\theta) > \frac{5}{9}\} \subset \{[(x,1,y)]_\sim : (x,y) \neq (1,1)\}$. Since $\max_{\theta\in G} \varsigma(\theta) = \frac{16}{45} < \frac{5}{9}$, we have $\mathcal{B} \subset \{[(x,1,y)]_\sim : (x-1)(y-1) > 0\}$.

To prove $\mathcal{B} \subset B_1 \cup B_2$, it suffices to prove

(1) If $\varsigma(t,t) > \frac{5}{9}$, then $t \in (-\infty, -4) \cup [0, \frac{1}{2}]$;

(2) $\varsigma(t,-4) < \frac{5}{9}$ for $t < -4$;

(3) $\varsigma(t, \frac{1}{2}) < \frac{5}{9}$ for $t \in [0, \frac{1}{2}]$.

Since $\varsigma(t,t) = \dfrac{D(t,t)}{|t-1|} = \dfrac{\max(|q_1(t) - 2q_2(t) + q_3(t)|, |3q_1(t) - q_2(t) - 1|)}{|t-1|}$. By applying Lemma 2.1, it can be shown that

$$\varsigma(t,t) = \frac{1 + q_2(t) - 3q_1(t)}{t-1}. \tag{2.22}$$

Combining with (2.3), one can prove that $\varsigma(t,t) > \frac{5}{9}$ implies $t \in (-\infty,-4) \cup [0, \frac{1}{2}]$. Thus (1) is proved.

For $t < -4$, we have $\varsigma(t,-4) = \dfrac{\max(D(t,-4), D(-4,t))}{1-t}$. One can verify that $\varsigma(t,-4) < \frac{5}{9}$ for $t < -4$. And (3) can be proved in the same manner. ■

Proof of [S2]: Let $\theta = [(x,1,y)]_\sim \in B_1$, i.e. $(x,y) \in (-\infty, 4]^2$. Since $\varsigma(x,y) = \varsigma(y,x)$, it follows from Lemma 2.3(i) and (2.22) that

$$\max_{(x,y)\in(-\infty,-4]^2} \varsigma(x,y) = \max_{\substack{(x,y)\in(-\infty,4]^2\\ y\leqslant x}} \varsigma(x,y) = \max_{x\in(-\infty,-4]} \varsigma(x,x) = \max_{x\in(-\infty,-4]} \frac{1+q_2(x) - 3q_1(x)}{x-1}.$$

Applying (2.3), it can he shown that

$$\max_{x\in(-\infty,-4]} \frac{1+q_2(x)-3q_1(x)}{x-1} = \frac{1 + q_2(-\frac{121}{25}) - 3q_1(-\frac{121}{25})}{-\frac{121}{25} - 1} = \frac{41}{73}.$$

Therefore $\max_{\theta\in B_1} \varsigma(\theta) = \varsigma(-\frac{121}{25}, -\frac{121}{25}) = \frac{41}{73}$. ■

A similar calculation establishes [S3].

Proof of [S5]: Suppose $\theta = [(x,1,y)]_\sim \in B_{21}$, then $x, y \in [\frac{3}{8}, \frac{1}{2}]$. From [S3], we have $\varsigma(\theta) \leqslant \frac{16}{27} < 1.35 \times \frac{5}{9}$. Therefore (2.18) is true for $N = 0$. It suffices to prove that for all $N > 0$,

$$\sum_{n=0}^{N} \ln(\varsigma(\varrho_2^n(\theta))) \leqslant (N+1)\ln(\frac{5}{9}) + \ln(1.35). \tag{2.23}$$

Without loss of generality, we can assume $y \geqslant x$.

Let $\bar{\theta} = [(x,1,x)]_\sim$. First, we are going to prove

$$\prod_{n=0}^{N} \varsigma(\varrho_2^n(\theta)) \leqslant \prod_{n=0}^{N} \varsigma(\varrho_2^n(\bar{\theta})) \tag{2.24}$$

for any integer $N \geqslant 0$.

Suppose $\varrho_2^n(\theta) = [(x_n, 1, y_n)]_\sim$ and $\varrho_2^n(\bar{\theta}) = [(\bar{x}_n, 1, \bar{y}_n)]_\sim$. Since $\varrho_2(B_{21}) \subset B_{21}$, we have $(x_n, y_n), (\bar{x}_n, \bar{y}_n) \in [\frac{3}{8}, \frac{1}{2}]^2$. We first prove $x_n \geqslant \bar{x}_n$ for all $n \geqslant 0$.

One can show that $\bar{x}_n = \bar{y}_n$. And it follows from (2.12) and (2.3) that

$$x_{n+1} = \frac{12x_ny_n - y_n}{12x_ny_n + 2x_n + 2y_n}, y_{n+1} = \frac{12x_ny_n - x_n}{12x_ny_n + 2x_n + 2y_n}, \bar{x}_{n+1} = \frac{12\bar{x}_n - 1}{12\bar{x}_n + 4}. \tag{2.25}$$

Thus if $y_n \geqslant x_n$ then $y_{n+1} \geqslant x_{n+1}$. Since $y_0 = y \geqslant x = x_0$, by induction we have $y_n \geqslant x_n$ for all $n \geqslant 0$. Therefore if $x_n \geqslant \bar{x}_n$ then

$$x_{n+1} = \frac{12x_n - 1}{12x_n + 2 + 2\frac{x_n}{y_n}} \geqslant \frac{12x_n - 1}{12x_n + 4} \geqslant \frac{12\bar{x}_n - 1}{12\bar{x}_n + 4} = \bar{x}_{n+1}.$$

The last inequality is due to the monotonicity of $\frac{12x-1}{12x+4}$ on the interval $[\frac{3}{8}, \frac{1}{2}]$. Since $x_0 = \bar{x}_0 = x$, by induction we have $x_n \geqslant \bar{x}_n$ for all $n \geqslant 0$.

From (2.22) and (2.3), we can get $\varsigma(t,t) = \frac{5}{36t(1-t)}$ for $t \in [\frac{3}{8}, \frac{1}{2}]$. Since $\frac{5}{36t(1-t)}$ is decreasing on $[\frac{3}{8}, \frac{1}{2}]$, we have $\varsigma(x_n, x_n) \leqslant \varsigma(\bar{x}_n, \bar{x}_n)$.

Since $y_n \geqslant x_n$, it follows from Lemma 2.3(i) that $\varsigma(\varrho_2^n(\theta)) = \varsigma(x_n, y_n) \leqslant \varsigma(x_n, x_n) \leqslant \varsigma(\bar{x}_n, \bar{x}_n) = \varsigma(\varrho_2^n(\bar{\theta}))$ for all $n \geqslant 0$. Therefore $\prod_{n=0}^{N} \varsigma(\varrho_2^n(\theta)) \leqslant \prod_{n=0}^{N} \varsigma(\varrho_2^n(\bar{\theta}))$ for all $N \geqslant 0$.

Let $g_n = \frac{1}{1/2 - \bar{x}_n}$, then it follows from (2.25) we get $g_{n+1} = \frac{5}{3}g_n - 2$ or $g_{n+1} - 3 = \frac{5}{3}(g_n - 3)$. So $g_n = (\frac{5}{3})^n(g_0 - 3) + 3$. Let $e_n = \frac{1}{g_n} = 1/2 - \bar{x}_n$, then $\varsigma(\varrho_2^n(\bar{\theta})) = \varsigma(\bar{x}_n, \bar{x}_n) = \frac{5}{36\bar{x}_n(1-\bar{x}_n)} = \frac{5}{9}\frac{1}{1-4e_n^2}$. Since $g_0 = \frac{1}{1/2 - x} \geqslant \frac{1}{1/2 - 3/8} = 8$, we have $g_n \geqslant 5(\frac{5}{3})^n + 3$. So $e_n = \frac{1}{g_n} \leqslant \frac{1}{5(\frac{5}{3})^n + 3}$. Thus

$$\varsigma(\varrho_2^n(\bar{\theta})) = \frac{5}{9}\frac{1}{1 - 4e_n^2} \leqslant \frac{5}{9}\frac{1}{1 - 4(\frac{1}{5(\frac{5}{3})^n + 3})^2} = \frac{5}{9}\frac{F_n}{F_n - 4} = \frac{5}{9}(1 + \frac{4}{F_n - 4}),$$

where $F_n = (5(\frac{5}{3})^n + 3)^2$. Then, by using the inequality $\ln(1+x) \leqslant x$, we have

$$\ln(\varsigma(\varrho_2^n(\bar{\theta}))) \leqslant \ln(\frac{5}{9}(1 + \frac{4}{F_n - 4})) = \ln(\frac{5}{9}) + \ln(1 + \frac{4}{F_n - 4}) \leqslant \ln(\frac{5}{9}) + \frac{4}{F_n - 4}.$$

So for $N > 0$,

$$\begin{aligned}
\sum_{n=0}^{N} \ln(\varsigma(\varrho_2^n(\bar{\theta}))) &\leqslant (N+1)\ln(5/9) + \sum_{n=0}^{N} \frac{4}{F_n - 4} \\
&= (N+1)\ln(5/9) + \sum_{n=0}^{N} \frac{4}{(5(\frac{5}{3})^n + 3)^2 - 4} \\
&= (N+1)\ln(5/9) + \sum_{n=0}^{N} (\frac{1}{5(\frac{5}{3})^n + 1} - \frac{1}{5(\frac{5}{3})^n + 5}) \\
&= (N+1)\ln(5/9) + \frac{1}{15} + \sum_{n=1}^{N} (\frac{1}{5(\frac{5}{3})^n + 1} - \frac{1}{5(\frac{5}{3})^n + 5}) \\
&\leqslant (N+1)\ln(5/9) + \frac{1}{15} + \sum_{n=1}^{N} (\frac{1}{5(\frac{5}{3})^n} - \frac{1}{5(\frac{5}{3})^{n+1}}) \\
&\leqslant (N+1)\ln(5/9) + \frac{1}{15} + \frac{1}{5(\frac{5}{3})} \\
&= (N+1)\ln(5/9) + \frac{14}{75} < (N+1)\ln(5/9) + \ln(1.35).
\end{aligned}$$

Thus $\sum_{n=0}^{N} \ln(\varsigma(\varrho_2^n(\theta))) \leqslant \sum_{n=0}^{N} \ln(\varsigma(\varrho_2^n(\bar{\theta}))) < (N+1)\ln(5/9) + \ln(1.35)$. ■

3 Remarks

The proofs of the end-point cases $p = 1, \infty$ of (1.3), being so similar, suggest that the same technique may be applied to prove the whole conjecture (1.3). The major impediment is that no closed-form expressions are available for the operator $S_{p,1,3}$ when $p \neq 1, 2, \infty$. However, it may be possible to make use of monotonicity and continuity properties in p, we leave this as future work.

Actually, for any dilation factor $d \geqslant 2$, closed-formed expressions for $S_{1,1,d}$ and $S_{\infty,1,d}$ are always available. The dyadic case $d = 2$ is studied closely by Oswald in [4, 3]. In principle our method can be used directly to attack the conjecture (1.2) in the end-point cases. One may think that by going dyadic instead of triadic and keeping other parameters unchanged, the problem should only get easier — *for us the situation is just the opposite.* In the dyadic case, the neighborhood factor of $S^{[1]}$ is 4, meaning, in particular, that the dynamical behaviors of two maps acting on $\mathbb{P}(\mathbb{R}^4)$ – as opposed to three maps acting on $\mathbb{P}(\mathbb{R}^3)$ in the triadic case – have to be studied. Being one dimension higher, it is hardly surprising that more tedious calculations get involved.

We quote from [2, Preface]:

"*My view is that overall we know too much about linear PDE and too little about nonlinear PDE.*"

While we would not complain that too much is known about linear subdivision, it is certainly the case that we know too little about nonlinear subdivision.

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