Approximation Order/Smoothness Tradeoff in Hermite Subdivision Schemes

Thomas P.-Y. Yu

Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, New York 12180-3590, USA.
Email: yut@rpi.edu

ABSTRACT

It is well-known to waveleticians that refinable functions exhibit subtle relationships between their approximation order and smoothness properties. We show how one can exploit this phenomenon to construct Hermite subdivision schemes with optimal smoothness but suboptimal approximation order for a given support size of the subdivision mask. The construction method considered here is based on a blend of the theory of subdivision schemes and computational techniques in non-smooth optimization. Our construction method produces schemes which are much smoother than those constructed based on optimizing approximation orders. We discuss also several interesting bivariate Hermite schemes, with appealing symmetry property, and illustrate how they can be applied to build interpolating subdivision surfaces.

Keywords: Subdivision Schemes, Refinable Functions, Wavelets, Hermite Subdivision, Subdivision Surfaces, Interpolation

1. INTRODUCTION

1.1. Approximation Order/Smoothness Tradeoff of Refinable Functions

Recall that a function vector is 2-refinable, or simply refinable, if it satisfies

$$\Phi = \sum_{\alpha} c_\alpha \Phi(2 \cdot -\alpha). \tag{1}$$

It is known to waveleticians that the *approximation order* and *smoothness* of a refinable function have a subtle relationship, except in very special cases (e.g. when ϕ is a spline, where the two properties go hand-in-hand.[1]) Under certain conditions, a C^k refinable function vector guarantees to provide approximation order $k+1$. The latter condition can be translated to the condition

$$\mathrm{span}\{\phi(\cdot - \alpha) : \phi \in \Phi,\ \alpha \in \mathbb{Z}^s\} \supseteq \Pi_k,$$

as well as to a simple set of linear conditions – usually referred to as **sum rules** – on the mask $(c_\alpha)_\alpha$ of the refinement equation (1).

In short, *smooth refinable functions provide good approximation orders*[2]; the converse implication, however, is far from the truth. The consequence of this comment is that when one is interested in optimal smoothness, then one really should not optimize approximation order. To put it in a different way, one can sacrifice some approximation orders for smoothness. The goal of this article is to discuss how one can capitalize on this observation to construct smooth Hermite interpolatory subdivision schemes and potentially other subdivision schemes.

Before we specialize to Hermite schemes, we note that this basic idea had been used in other settings, notably by Daubechies and also by Dyn *et al.* In these applications, however, the number of parameters is quite small (one or two) and relatively ad-hoc optimization methods suffice. For example, in Daubechies' work[3] a easily computable upper bound, rather than the honest Hölder exponent, was optimized and it happened that a smoother wavelet was discovered. One needs a more careful optimization method when dealing with our parametric Hermite subdivision masks, which typically have many more than one parameter (see Section 3.3 and Table 1.)

1.2. Hermite and Vector Subdivision Schemes

The refinable functions considered in this paper will be those generated by *vector subdivision* algorithms. In fact, we will mostly be working on a special case of subdivision schemes known as Hermite interpolatory subdivision, which the latter will be defined formally in the next section. In the wavelet literature, vector subdivision schemes are related to the so-called *multiwavelets*. We are interested in vector subdivision here for exactly the same reason waveleticians are interested in multiwavelets: improved design flexibility and better smoothness/support size tradeoff. The main difference is that our interest is more geared towards applications in subdivision surface design, where smoothnness is typically more important than approximation order, and one is interested in keeping the support size small due to the presence of *extraordinary vertices*. We will show, for example, how one can construct a bivariate Hermite subdivision scheme with multiplicity 3, with a much smaller support than the well-known Butterfly scheme and, yet, possesses the same smoothness as the Butterfly scheme. Thanks to its small support, this scheme also happens to have a remarkable property when applied to the setting of irregular triangular meshes with extraordinary vertices (Figure 2.)

The flexibility of Hermite schemes is reflected by the fact that their masks, even when subjected to various constraints, having many degrees of freedom. We will show how one can exploit these degrees of freedoms to optimize smoothness.

1.3. Organization

Section 2 formally discusses Hermite schemes and the criteria for their convergence and smoothness, generalizing the original proposal of Merrien.[4] Section 3, the longest section in this paper, shows how in the univariate case the smoothness optimization can be carried out using a blend of the theory of subdivision schemes and techniques in optimization. Section 4 discusses extension to the multivariate case. The same section also gives examples on application of a bivariate Hermite subdivision schemes to subdivision surfaces.

2. HERMITE INTERPOLATORY SUBDIVISION

We use the standard multi-index notations $\theta = (\theta_1, \ldots, \theta_s)$ and $|\theta| = \sum_{i=1}^{s} |\theta_i|$, $z^\theta = z_1^{\theta_1} \cdots z_s^{\theta_s}$. The dictionary ordering is defined by: $\theta < \theta' \iff \exists i \in \{1, \ldots, s\}$ s.t. $\theta_i < \theta'_i$ and $\theta_k \leq \theta'_k$ $\forall k > i$. Unless otherwise stated, the notation θ is reserved for s-vector of non-negative integer entries. Denote by $\{e_1, \ldots, e_s\}$ the standard ordered basis of $\mathbb{R}^s$.

In the regular grid setting, a stationary subdivision operator is one of the form $S : [l(\mathbb{Z}^s))]^m \to [l(\mathbb{Z}^s)]^m$,

$$(Sv)_\alpha = \sum_{\beta \in \mathbb{Z}^s} A_{\alpha - 2\beta} v_\beta \tag{2}$$

where $(A_\alpha)_\alpha$ is a compactly supported sequence of $m \times m$ matrices, usually referred to as the *mask* of the subdivision scheme. It is called a scalar and a vector scheme when $m = 1$ and $m > 1$, respectively.

(Since the mask is compactly supported, one is free to consider S as a bounded operator on any one of the Banach spaces $[l^p(\mathbb{Z}^s)]^m$, $p \in [1, \infty]$.)

We intend to explore here a specific class of (2) known as Hermite interpolatory schemes. There are different ways to define such schemes, especially if we choose to work with the *refinement equation* :

$$F(x) = \sum_{\beta \in \mathbb{Z}^s} A_\beta^T F(2x - \beta)$$

associated with the subdivision operator (2), which the former is more relevant to wavelet constructions. Since we are primarily interested in subdivision algorithms, we choose the following definition adapted from.[5]

Fix differentiation order $d \geq 0$ and dimension $s \geq 1$. The integer $m = \binom{d+s}{s}$ will become the multiplicity of a Hermite scheme. Define the Hermite data sampler $H_{j,\beta} : C^d(\mathbb{R}^s) \to \mathbb{R}^m$ at dyadic points $x_{j,\beta} = 2^j \beta$ by:

$$(H_{j,\beta} f)_\theta := \frac{\partial^{|\theta|}}{\partial^{\theta_1} x_1 \cdots \partial^{\theta_s} x_s} f(x_{j,\beta}) \times 2^{-j|\theta|} \qquad |\theta| \leq d.$$

In this article we will mostly be dealing with the case of $s = 2$, $d = 1$.

Definition 2.1. *Fix differentiation order $d \geq 0$ and dimension $s \geq 1$. A* **Hermite interpolatory subdivision scheme reproducing** Π_k *is a subdivision operator of the form (2) which satisfies the following property: For any $p \in \Pi_k$, $j \geq 0$,*

$$v_\beta = H_{j,\beta}\, p \quad \textit{implies} \quad (Sv)_\beta = H_{j+1,\beta}\, p. \tag{3}$$

Two important comments are in order:

1. Routine calculation shows that if S is such that (3) holds for $j = 0$, then it holds also for all j. Furthermore, by linearity and shift invariance of subdivision (3) boils down to a finite set of linear relationships on the non-zero entries of the mask $(A_\alpha)_\alpha$ of S. Specifially, the interpolatory properties imposed by (3) (note $x_{j+1,2\beta} = x_{j,\beta}$) implies that the even taps of the mask must satisfy

$$A_{2\alpha} = M_m(2^{-1})\delta_{\alpha,\mathbf{0}} \tag{4}$$

where M_m is the diagonal matrix defined by $(M_m(h))_{\theta,\theta'} = h^{|\theta|}$ if $\theta = \theta'$ and 0 otherwise (here rows and columns of $M_m \in \mathbb{R}^m$ are indexed according to the dictionary order of θ and θ'.) For the "odd" taps, one gets a genuine linear system: collect the nonzero non-even entries of the mask into a column vector a, then (3) amounts to a linear system

$$\Gamma a = b. \tag{5}$$

We take the point of view that we are in search for a "good" subdivision mask; and thus a is viewed as an *unknown* vector. The length of a (= number of unknowns in (5)) is dependent on *support size* of the mask $(A_\alpha)_\alpha$. On the other hand, the number of equations in (5) is determined by the desired polynomial reproduction degree k.

We call any vector subdivision scheme (2) whose mask obeys (4) a **Hermite interpolatory subdivision scheme**.

2. In the 1-D case $s = 1$, it is shown in[5] that any C^r-convergent Hermite interpolatory subdivision scheme (according to Definition 2.3 below) necessarily reproduces Π_k. A similar theorem is proved[6] in cases of $m = 1$ (i.e. $d = 0$) and s arbitrary. We expect that a similar theorem holds for arbitrary s and d. We make these results explicit in the following theorem. Related results can be found also in work of Jia.[7]

Theorem 2.2. *For $s = 1$, d arbitrary, or s arbitrary, $d = 0$, an s-dimensional Hermite interpolatory scheme S with differentiation order d is C^k convergent (see Definition 2.3) only if S reproduces Π_k.*

Our criterion for convergence and smoothness of Hermite schemes is defined by:

Definition 2.3. *Let $k \geq d$. A Hermite subdivision scheme is C^k-convergent if for any $f \in [l_0(\mathbb{Z}^s)]^m$, there exists $F \in C^k(\mathbb{R}^s)$ such that*

$$H_j F = S^j f, \qquad \forall j \in \mathbb{N}. \tag{6}$$

More generally, let B be a Banach space such that $B \subset C^d(\mathbb{R}^s)$, A Hermite subdivision scheme is B-convergent if for any $f \in [l_0(\mathbb{Z}^s)]^m$, there exists $F \in B$ such that $H_j F = S^j f$, $\forall j \in \mathbb{N}$.

To measure smoothness we use membership in generalized Lipschitz space $Lip^*(\nu, L_p(\mathbb{R}^s))$. In fact we are mainly interested in the cases $p = \infty$ — closely related to Hölder spaces $C^\nu(\mathbb{R}^s)$ — and $p = 2$ — closely related to L_2-Sobolev spaces $W_2^\nu(\mathbb{R}^s)$.

Definition 2.4. *For a C^d-convergent Hermite scheme S of differentiation order d,*

$$\nu_p(S) := \sup\{\nu : S \textit{ is } Lip^*(\nu, L_p(\mathbb{R}^s))\textit{-convergent}\}.$$

3. UNIVARIATE CASE

It is well-known that the key idea in the celebrated Dubuc-Deslauriers schemes[8,9] can be adapted to build wavelets and subdivision schemes in various settings.[10–15] In this approach, one performs local polynomial interpolation of data at one scale and evaluate the interpolant in order to predict data at the next finer scale.[15]

The Dubuc-Deslauriers recipe can be used to build 1-D Hermite interpolatory scheme in a very straightforward manner.[15] In the notations we have already set up earlier, it amounts to:

1. Choose an integer $L > 0$ and differenatiation order $d(= m-1)$. Consider Hermite subdivision masks supported at $[-(2L-1),(2L-1)]$. According to (4) of comment (1) above, the only unknowns in the mask are

$$\{A_\alpha : \alpha \in [-(2L-1),(2L-1)], \alpha \text{ odd}\}. \tag{7}$$

2. To solve for the mask entries (7), we invoke the necessary C^k-convergence condition in comment (2) above: (A_α) must satisfy (3) for all $p \in \Pi_k$. By well-posedness of Hermite-interpolation in 1D, the associated linear system (5) has a unique solution if we choose $k = 2mL - 1$.

We refer to the so obtained 1-D Hermite interpolatory subdivision scheme as

$$\textsc{Hermite}(m, L, 2mL-1).$$

We note that no Hermite interpolatory schemes with mask supported on $[-(2L-1),(2L-1)]$ can reproduce Π_l for $l \geq 2mL$, thus $\textsc{Hermite}(m, L, 2mL-1)$ is the scheme that *optimizes* approximation order.

Note. Although we do not make it explicit here, there is a direct connection between the highest degree of polynomial reproducibility of a subdivision scheme and the approximation order of the shift-invariant subspace generated by the refinable functions associated with the subdivision scheme. We will therefore use the terms **approximation order** and **degree of polynomial reproducibility** interchangeably.

3.1. $\textsc{Hermite}(m, L, 2mL-1)$, $k < 2mL-1$

We now come back to the main theme of this article: approximation order/smoothness tradeoff.

In[16] the author set up the necessary computational tools for calculating critical Hölder regularity ($\nu_\infty(S)$) of 1-D Hermite schemes. Results there show that when $S = \textsc{Hermite}(2, 2, 7)$,

$$\nu_\infty(S) = 3. \tag{8}$$

In fact for almost all initial sequences the subdivision limits are only C^2 with 2nd derivatives lying in $C^{1-\epsilon}$ for any $\epsilon > 0$. This result exemplifies the claim that Π_k reproduction is far from a sufficient condition for C^k-convergence.

To obtain smoother schemes, we first consider masks which do *not* optimize approximation order. We denote by

$$\textsc{Hermite}(m, L, k, \textsc{Assymmetric}) \tag{9}$$

the set of all Hermite interpolatory subdivision schemes $S = S_A$ with multiplicity m, mask $(A_\alpha)_\alpha$ supported at $[-(2L-1),(2L-1)]$, and reproduce Π_k. Under these conditions, one sees that the associated linear system (5) has $2Lm^2$ unknowns and $(k+1)m$ linear equations. The linear system is of full rank, owing to well-posedness of 1-D Hermite-interpolation.[16] Thus the subdivision masks in $\textsc{Hermite}(m, L, k, \textsc{Assymmetric})$ can be linearly parametrized by $2m^2L - m(k+1)$ parameters.

In practice, one is more interested in schemes which are **symmetric**. A 1-D Hermite scheme is symmetric if its mask satisfies

$$A_{-k} = NA_kN^{-1} \quad \text{where} \quad N = \text{diag}((-1)^0, (-1)^1, \ldots, (-1)^{m-1}). \tag{10}$$

Reason for this definition: If the application of a symmetric Hermite scheme to an initial sequence $f_{0,k}$ converges to $F \in C^s$, $s \geq (m-1)$, then the same scheme, when applied to the initial data $(g_{0,k})_l := G^{(l)}(k)$, $G := F(-\cdot)$, converges to G.

We denote the symmetric members in (9) by $\textsc{Hermite}(m, L, k, \textsc{Symmetric})$ or simply

$$\textsc{Hermite}(m, L, k) \ (\subset \textsc{Hermite}(m, L, k, \textsc{Asymmetric}))\,.$$

To solve for the masks in $\textsc{Hermite}(m, L, k)$ one needs to solve the linear relations (5) **and** (10). An interesting point here is that these two sets of linear relations are *not* independent. Calculations in[16] show that when $k \le 2mL - 1$, solution exists and the nullity of the linear system is $m^2L - \lceil m(k+1)/2 \rceil$.

As an example, masks in $\textsc{Hermite}(2, 2, 4)$ can be represented by 3 parameters:

$$A_1 = \begin{bmatrix} 63/128 - 3\,t_1 & 9/64 + 3\,t_1 \\ -3/4 + 9\,t_2 - 24\,t_3 & -1/8 + 6\,t_2 - 13\,t_3 \end{bmatrix},\ A_3 = \begin{bmatrix} 1/128 + 3\,t_1 & t_1 \\ t_2 & t_3 \end{bmatrix}. \tag{11}$$

The techniques discussed in the next subsections will be applied to show that by suitably choosing (t_1, t_2, t_3) in this mask, one can obtain C^4-convergent schemes.

3.2. A Smoothness Formula

Let $\{f_{j,k} : j > 0, k \in \mathbb{Z}\}$ be generated by a particular scheme $S \in \textsc{Hermite}(m, L, D)$ with initial (Hermite) data $\{f_{0,k} : k \in \mathbb{Z}\}$. For convenience, we will only consider the case of

$$D \ge m - 1.$$

Each $f_{j,k}$ is a column vector in $\mathbb{R}^m$, indexed by $l = 0, \ldots, (m-1)$. Intuitively, $f^l_{j,k}$ is interpreted as $2^{-jl}F^{(l)}(x_{j,k})$ for some $F \in C^{m-1}$. Notice it is solely for notational convenience that we assume that such a function F exists. In fact the following definitions of $u^{[r]}$ and $\triangle u^{[r]}$ do not depend on F, but only on $f_{j,k}$.

Let

$$t_{j,n} = 2^{-j}\left\lfloor \frac{n}{m} \right\rfloor.$$

For $m - 1 \le r \le 2mL$, define $u^{[r]}_{j,k} \in \mathbb{R}^m$ by

$$\left(u^{[r]}_{j,k}\right)_l = [t_{j,km+l}, \ldots, t_{j,km+l+r}]F, \qquad l = 0, \ldots, (m-1). \tag{12}$$

These are so-called **multiple-knot divided differences**. For example, when $r = m - 1 = 2$, one has

$$u^{[m-1]}_{j,k} = [[x_{j,k}, x_{j,k}, x_{j,k}]F,\ [x_{j,k}, x_{j,k}, x_{j,k+1}]F,\ [x_{j,k}, x_{j,k+1}, x_{j,k+1}]F]^T\,.$$

We will denote the operator that maps $\{f_{j,k}\}_k$ to $\{u^{[r]}_{j,k}\}_k$ by ∂^r and write

$$u^{[r]}_{j,\cdot} = \partial^r f_{j,\cdot}.$$

Define the forwarded differencing sequence of $u^{[r]}_{j,k}$ by

$$(\triangle u^{[r]}_{j,k})_l = \begin{cases} (u^{[r]}_{j,k})_{l+1} - (u^{[r]}_{j,k})_l & 0 \le l < m-1, \\ (u^{[r]}_{j,k+1})_0 - (u^{[r]}_{j,k})_{m-1} & l = m-1. \end{cases} \tag{13}$$

According to the definition of divided difference, $\triangle u^{[r-1]}_{j,k}$ and $u^{[r]}_{j,k}$ are related by

$$(u^{[r]}_{j,k})_l = (\triangle u^{[r-1]}_{j,k})_l/(2^{-j}C(l; m, r)), \tag{14}$$

where $C(l; m, r) = \lfloor \frac{l+r}{m} \rfloor$.

Dyn and Levin showed in[5] that for any Hermite-interpolating subdivision scheme that reproduces Π_D, $\{u^{[r]}_{j,k}\}$, $r = (m-1), \ldots, (D+1)$ and $\{\triangle u^{[r]}_{j,k}\}$, $r = (m-1), \ldots, D$ follow vector subdivision schemes with compactly supported masks. Translating their results to our setting, we have

- for $(m-1) \le r \le D+1$, a finitely supported matrix sequence $\{M_l^{[r]}\}_l$ such that

$$u_{j+1,k}^{[r]} = \sum_l M_{k-2l}^{[r]} u_{j,l}^{[r]}; \quad \text{and} \tag{15}$$

- for $(m-1) \le r \le D$, a finitely supported matrix sequence $\{D_l^{[r]}\}_l$ such that

$$\triangle u_{j+1,k}^{[r]} = \sum_l C_{k-2l}^{[r]} \triangle u_{j,l}^{[r]}. \tag{16}$$

[5] also provides formulas for computing the matrices $\{M_l^{[r]}\}$ and $\{D_l^{[r]}\}$ from the mask $\{A_l\}$ of a Hermite scheme. We will generally denote by $S^{[r]}$ the subdivision scheme of $\{u_{j,k}^{[r]}\}$ and by $\triangle S^{[r]}$ that of $\{\triangle u_{j,k}^{[r]}\}$. We call $\{M_l^{[r]}\}$ and $\{D_l^{[r]}\}$ the subdivision *masks* of $S^{[r]}$ and $\triangle S^{[r]}$ respectively. Following our earlier notation, we have:

$$S_M = S^{[r]} \quad \text{and} \quad S_D = \triangle S^{[r]}.$$

The article[16] show how one can use the mask D to compute critical Hölder regularity $\nu_\infty(S)$ of a Hermite scheme S_A with mask A. The basic result is that if $S = S_A$ reproduces Π_k but not Π_{k+1}, and if $S_D = S^{[k]}$ then:

$$\nu_\infty(S_A) = k - \log_2(\rho_\infty(S_D)) \tag{17}$$

where for $p \in [1,\infty]$, $\rho_p(S_D)$ is the spectral radius of S_D when viewed as a bounded operator from $[l_p(\mathbb{Z})]^m$ to itself.

For any subdivision operator S (2) it is well-known that there is a convenient way to compute $||S^j||_{[l_\infty(\mathbb{Z})]^m \to [l_\infty(\mathbb{Z})]^m}$; hence to compute (17) one can use Gelfand's spectral radius formula $\rho(S) = \lim_{j\to\infty} ||S^j||^{1/j}$.

However, the resulted iterative computational method has a computational and memory complexity both of $O(2^j)$, where j is the number iterations. Worse, it is observed that the rate of convergence of this method is usually extremely slow. (It is mostly an empirical observation. But in certain cases, one can prove $\rho(S) - ||S^j||^{1/j} \asymp 1/j$.)

One can remedy the situation by replacing $p = \infty$ with $p = 2$. Using the elementary fact that $||\cdot||_{l_\infty} \le ||\cdot||_{l_2}$ one can lower bound (17) by

$$\nu_\infty(S_A) \ge k - \log_2(\rho_2(S_D)). \tag{18}$$

This is effectively a Sobolev embedding theorem in action, although it will require extra arguments to make this connection precise.

The righthand side (18) equals the spectral radius of a finite matrix. Consider the transition operator $T = T_D : [l_\infty(\mathbb{Z})]^{m\times m} \to [l_\infty(\mathbb{Z})]^{m\times m}$,

$$T_D w(\alpha) = \frac{1}{2} \sum_{\beta,\gamma\in\mathbb{Z}} D^*(2\alpha-\beta) w(\beta+\gamma) D(\gamma). \tag{19}$$

Based on a shift, we can assume without loss of generality that the subdivision mask $(D_\alpha)_{\alpha\in\mathbb{Z}}$ is supported at $[0,\ldots,N_D]$. Under this assumption, one observes that the finite dimensional space

$$V := [l([-N_D, N_D])]^{m\times m} \tag{20}$$

(the space of all matrix sequences supported at $[-N_D, N_D]$) is an invariant subspace of T_D. In fact, it is well-known that

$$\rho_2(S_D) = \sqrt{2\rho(T_D|_V)}. \tag{21}$$

Thus we have the following easily computable bound

$$\nu_\infty(S_A) \ge k - \log_2 \sqrt{2\rho(T_D|_V)} \tag{22}$$

where $T_D|_V$ can be represented by a matrix of size $(2N_D+1)m^2$.

Note. We mention a subtle point here. There is already a vast literature on analysis of subdivision schemes and refinable functions; and many general smoothness results have been obtained. Thus it may seem unnecessary to use the specialized method here. For Hermite schemes, however, we have the multi-scale interpolatory property (6) as part of our requirements for convergence. The use of multiple-knot divided differences, besides being computationally effective,[16,17] seems indispensable for establishing condition (6). We refer to the paper[17] for technical details.

3.3. Smoothness Optimization

Since the masks of $\textsc{Hermite}(m, L, k)$ depend linearly on several parameters, one easily observes that the entries of $M = M(\mathbf{t})$, the matrix representation of $T_D|_V$ in the standard basis, depend quadratically on these parameters. Here $\mathbf{t} \in \mathbb{R}^N$, $N = m^2 L - \lceil m(k+1)/2 \rceil$ is the parameter vector.

Fix m and L, our goal here is to find the smoothest possible scheme among all of $\textsc{Hermite}(m, L, k)$, for different k. Of course, we have

$$\textsc{Hermite}(m, L, 0) \supset \cdots \supset \textsc{Hermite}(m, L, m-1) \supset \cdots \supset \textsc{Hermite}(m, L, 2mL-1).$$

With a chosen k (we consider only $k \geq D = m - 1$), one can first construct the parametric mask of $A = A(\mathbf{t})$ of $S \in \textsc{Hermite}(m, L, k)$, then form the mask $D = D(\mathbf{t})$ of $\triangle \partial^k S$. To compute the radius (21) at a specific set of parameter values, one can form the matrix $M(\mathbf{t})$ explicitly and then use a standard eignevalue solver. A more memory efficient method is to code up a routine for (19) and then use a matrix-free power method.[18]

To optimize smoothness, we are interested in solving, for $D \leq k < 2mL - 1$,

$$\min_{\mathbf{t} \in \mathbb{R}^N} \rho(M(\mathbf{t})). \tag{23}$$

Problems of the form (23) are non-smooth, non-convex optimization problems and may have many different local minima. In general, one can only ask for methods which can effectively locate local minima, and apply such a method a number of times to get an idea of the global minimum. Currently, two sets of solvers are available for this problem:

- Traditional unconstrained optimization methods such as Nealder-Mead type simplex search method or the Quasi-Newton method based on Broyden-Fletcher-Goldfard-Shanno update.[19] These methods are designed for optimizing smooth objective functions and, in principle, are not suitable for solving (23). However, these solvers are conveniently available in, for example, Matlab's optimization toolbox; and can be used without even providing gradient values

$$\frac{\partial \rho(M(\mathbf{t}))}{\partial t_i} \tag{24}$$

  of the objective function.

- A recently developed gradient-bundle method,[20] originally designed for solving the spectral abscissa optimization problem in control theory, can be adapted to optimize spectral radius. The resulted solver requires (24) as input; and strategically selects search directions at places where the objective function is non-smooth. In our case, since the entries of $M(\mathbf{t})$, being quadratic functions of the parameters, are smooth in $\mathbf{t}$, (24) exists almost everywhere and, moreover, can be analytically computed with a certain application of chain rule.[21]

We intend to discuss these two methods at a more technical level elsewhere. Here we report some numerical results, based on the first method above.

We focus on the case $(m, L) = (2, 2)$, the software tools discussed below allow one to explore other cases if wanted. Owing to (8), we aim for schemes in $\textsc{Hermite}(2, 2, k)$ which are at least C^3-convergent. This means, by Theorem 2.2, we have to search in $\textsc{Hermite}(2, 2, 3)$. The Matlab routines `MakeHermiteMask.m` & `DeriveHermiteMask.m`, based on symbolic computation and available at `http://www.rpi.edu/~yut/Hermite1.html`, can be invoked with the calling sequences

```
>> A = MakeHermiteMask(2,2,3);
>> D = DeriveHermiteMask(A,3);
```

to produce the parametric mask $A(\mathbf{t})$ of $S \in \textsc{Hermite}(2, 2, 3)$ and the mask $D(\mathbf{t})$ of $\triangle \partial^3 S$ respectively. The routine `TransitionOperator.m`, available at the same URL, can be used to produce a matrix representation $M(\mathbf{t})$ of $T_D|_V$, via the following calling sequence:

```
>> M = TransitionOperator(D);
```

Here the masks `A` and `D` have support lengths 7 and 5 respectively, with $N = 4$ parameters. By (20), `M` has size 36 by 36. With a change of format of M, one can then utilize the unconstrained optimization solver `fminu` in Matlab to solve (23). In this case, the minimum value 0.25 is found, proving that there exists a scheme in HERMITE$(2, 2, 3)$ with $\nu_\infty(S) \geq 3 - \log_2 \sqrt{2 \times 0.25} = 3.5$.

Based on certain facts related to L^2-Sobolev exponents (not discussed here), the above computational result actually hints to the possibility of a C^4-scheme which, by Theorem 2.2, can only be found in HERMITE$(2, 2, 4)$. A similar computation with $k = 4$ (in this case, support length of `D` $= 6$, # of parameters $= 3$, size of `M` $= 44$ by 44) shows that when

$$t_1 = 0.00853,\ t_2 = -0.0336,\ t_3 = -0.00847 \tag{25}$$

we have

$$\rho(M(\mathbf{t})) = 0.03096 \Longrightarrow \nu_\infty(S(\mathbf{t})) \geq 4 - \log_2 \sqrt{2 \times 0.03096} = 4.3458.$$

To conclude, the Hermite scheme with mask (11) and parameter values (25) yields limit curves with 2 more differentiability orders compared to those generated by the scheme Hermite$(2, 2, 7)$.

Before leaving this section, we mention that one may not always be able to gain smoothness by sacrificing approximation orders, this happens typically when $m = 1$, L small, or when $L = 1$. For example, the Deslauriers-Dubuc scheme based on local cubic polynomial interpolation (=HERMITE$(1, 2, 3)$) simultaneously optimizes smoothness and approximation order among all schemes in HERMITE$(1, 2, k$, ASSYMMETRIC$)$.[16]

4. BIVARIATE SCHEMES AND APPLICATION TO SUBDIVISION SURFACES

Our motivation for developing the computational optimization methods here is ultimately motivated by application in surface modelling based on subdivision surfaces. Firstly, smoothness (and other criteria pertaining to surface quality) is usually more important than approximation order in this application. Secondly, for various reasons the Deslauriers-Dubuc idea is much less natural in two or higher dimensions. Thirdly, Hermite schemes occur to be an interesting family of schemes because of their superior regularity when compared to their scalar counterparts with the same support sizes. Of course one can gain regularity by enlarging the support size of the scheme. In subdivision surfaces, however, it is important to keep the support size small in order to avoid the need to create many extra subdivision rules at the vicinity of **extraordinary vertices**.

We shall keep the exposition very brief in this section; and refer to the papers[17,22,23] for complete details.

4.1. Bivariate Schemes

In 1-D, a Hermite scheme, symmetric or not, is uniquely specified by a linear rule that predicts Hermite data at scale $j + 1$ dyadic point $x_{j+1,2\alpha+1}$ based on scale j Hermite data at $x_{j,\alpha+l}$, $l = -(L - 1), \ldots, 0, \ldots, L$. The rule is constant in j and in k. In 2-D, one in principle has to construct $2^2 - 1 = 3$ rules for the prediction of Hermite data at positions of the form

$$x_{j+1,(2\alpha_1+1,2\alpha_2)}, \quad x_{j+1,(2\alpha_1,2\alpha_2+1)}, \quad x_{j+1,(2\alpha_1+1,2\alpha_2+1)}. \tag{26}$$

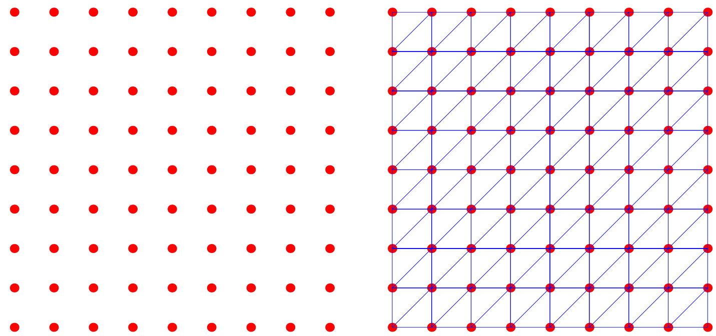
We will see below that a natural symmetry requirement reduces these three rules into one.

If one identifies $\mathbb{Z}^2$ with the regular 3-directional triangular grid in the plane, one naturally demands the scheme to possess the following symmetry property. Consider the group of all affine transformation $\mathcal{A}$ that leave the three-directional grid invariant. Because of the convolutional structure of subdivision, it suffices to consider the quotient group $\mathcal{D} := \mathcal{A} \backslash \mathcal{T}$, where $\mathcal{T} \subset \mathcal{A}$ is the (normal) subgroup of all integer translations. $\mathcal{D}$ is then isomorphic to the so-called dihedral group D_6 which contains 12 elements. With that in mind, we can simply regard $\mathcal{D}$ as the group of 12 matrices $\{I, R, R^2, R^3, R^4, R^5, F, FR, FR^2, FR^3, FR^4, FR^5\}$ where

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}. \tag{27}$$

If $f \in C^1(\mathbb{R}^2)$ and $g = f(A\cdot)$, $A \in \mathcal{D}$, then

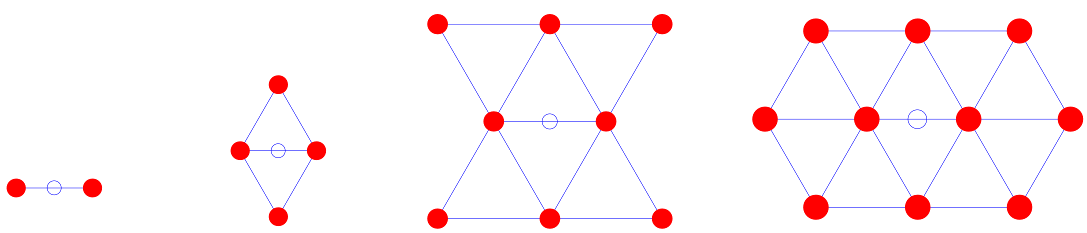
$$[g(\mathbf{x}), g_x(\mathbf{x}), g_y(\mathbf{x})]^T = \begin{pmatrix} 1 & 0 \\ 0 & A^T \end{pmatrix} [f(A\mathbf{x}), f_x(A\mathbf{x}), f_y(A\mathbf{x})]^T. \tag{28}$$



A bivariate Hermite subdivision scheme of differentiation order $d = 1$ (multiplicity $m = 3$) is symmetric with respect to D_6 if it preserves (28) in every subdivision step: Given any $F \in C^1$ and $A \in \mathcal{D}$. Let $G = F(A\cdot)$, we require that whenever $f_{j,\alpha} = [F(x_{j,\alpha}), F_x(x_{j,\alpha}) \times 2^{-j}, F_y(x_{j,\alpha}) \times 2^{-j}]^T$ and $g_{j,\alpha} = [G(x_{j,\alpha}), G_x(x_{j,\alpha}) \times 2^{-j}, G_y(x_{j,\alpha}) \times 2^{-j}]^T$, then application of one step of subdivision results in

$$g_{j+1,\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & A^T \end{pmatrix} f_{j+1,\alpha}, \quad \forall \alpha.$$

As in the 1-D case, symmetry imposes certain linear conditions on the subdivision mask $(A_\alpha)_\alpha$. Together with polynomial reproduction conditions (3), one arrives at, again similar to the 1-D case, a linear system of equations for the mask entries. We recall that the number of unknowns in such a linear system is dependent on the chosen support size of the mask, which the latter can also be determined by an *interpolation stencil*. Interpolation stencil here refers to the spatial configuration of scale j Hermite data used to predict the Hermite data at a scale $j + 1$ dyadic point of one of the forms (26). Note that when symmetry is required, only stencils with certain symmetric shape are allowed. Figure 1 gives examples of such stencils. It is easy to see that our symmetry requirement implies



(a) Two-Point (b) Diamond (c) Butterfly (d) Ten-Point

Figure 1. Symmetric Interpolation Stencils

that the stencils for the three types of points in (26) must share the same shape.

Unlike the 1-D case, the rank of the above mentioned linear systems seem much harder to analyze. Fortunately, one can make use of symbolic algebra software to solve these linear systems and also determine their nullity. See Table 1.

For a more formal and general description of symmetry, with respect to dilation matrices other than $2I_{2\times2}$, see.[22] Dilation matries with determinant $= 2, 3$ are of particular interests in applications, because the resulted subdivision schemes expand data at much slower rates and hence allow more subdivision steps.

The regularity analysis and smoothness optimization computation are much tricker in the multivariate case. The major issue is that the factorization technique used in 1-D can no longer be conveniently employed when the spatial dimension $s > 1$; and one has to use a different technique.[17]

We discuss in more details the two-point scheme with approximation order $k = 2$; the mask of this scheme is

k		0	1	2	3	4	5	6
Two-Point	5	4	2	1	n.s.	n.s.	n.s.	n.s.
Diamond	10	9	7	5	1	n.s.	n.s.	n.s.
Butterfly	19	18	16	14	10	7	1	n.s.
Ten-Point	24	23	21	19	15	12	6	3

Table 1. Number of parameters (=nullity in the linear system determining the parametric mask) in the symmetric Hermite subdivision mask reproducing Π_k and with support size determined by each of the interpolation stencils in Figure 1. The first column corresponds to subdivision masks which satisfy only symmetry but no polynomial reproducibility of any degree.

found to be:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(-1,1)}, \quad \begin{bmatrix} 1/2 & 0 & 1/8 \\ t & 1/4 & t/2 \\ -2t & 0 & 1/4-t \end{bmatrix}_{(0,1)}, \quad \begin{bmatrix} 1/2 & 1/8 & 1/8 \\ -t & 1/4-t/2 & -t/2 \\ -t & -t/2 & 1/4-t/2 \end{bmatrix}_{(1,1)},$$

$$\begin{bmatrix} 1/2 & -1/8 & 0 \\ 2t & 1/4-t & 0 \\ -t & t/2 & 1/4 \end{bmatrix}_{(-1,0)}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}_{(0,0)}, \quad \begin{bmatrix} 1/2 & 1/8 & 0 \\ -2t & 1/4-t & 0 \\ t & t/2 & 1/4 \end{bmatrix}_{(1,0)},$$

$$\begin{bmatrix} 1/2 & -1/8 & -1/8 \\ t & 1/4-t/2 & -t/2 \\ t & -t/2 & 1/4-t/2 \end{bmatrix}_{(-1,-1)}, \quad \begin{bmatrix} 1/2 & 0 & -1/8 \\ -t & 1/4 & t/2 \\ 2t & 0 & 1/4-t \end{bmatrix}_{(0,-1)}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(1,-1)}.$$

Bin Han pointed out to the author that when $t = 1/2$, the scheme is actually a spline scheme; yielding C^1 piecewise quadratic polynomials. At the time this article is written, the author sees very strong evidence that this scheme is a subdivision realization of the well-known Powell-Sabin scheme C^1 finite-elements. Resolution of this speculation will be reported in.[22]

4.2. Hermite Subdivision Surfaces

Bivariate Hermite subdivision schemes, constructed in the regular grid setting, can be used to produce 2-surfaces of arbitrary topology. In fact it is a common practice in subdivision surface to first construct a smooth subdivision schemes on a regular grid, and then construct and analyze extraordinary rules used in the vicinity of extraordinary vertices.[24–26] For Hermite schemes, one needs also to construct a rule to introduce gradient vectors when a simplicial complex with only vertex positions are given.

Such a preconditioning rule, as well as other implementational details of Hermite schemes on general triangular meshes, are discussed in detail in the report.[23] Figures 2-3 present some of our results. We compare our 2-point Hermite schemes (presented in the the previous section) with the Butterfly scheme[27] and the modified Butterfly scheme.[28] Away from extraordinary points, all three schemes produce almost C^2 parametrizations. At this point, the regularity properties of Hermite schemes at extraordinary vertices are not yet well-understood; extending the analysis tools for scalar schemes[24–26] to their vector counterparts would be a natural point of departure.

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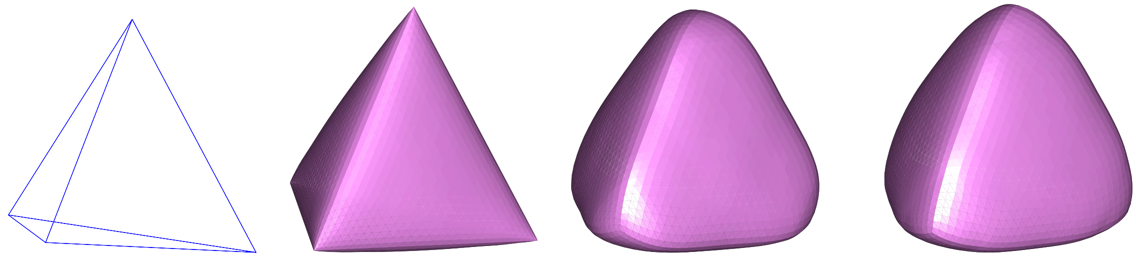

(a) Tetrahedron (b) Butterfly Scheme (c) Modified Butterfly Scheme (d) Two-Point Hermite Scheme (with no extraordinary vertex rules)

Figure 2. *A surprise of Hermite scheme.* the 2-Point Hermite scheme discussed in the previous section, when applied to arbitrary triangular meshes without introducing any extraordinary rules, produces surfaces as smooth as those produced by the modified Butterfly scheme, at and away from extraordinary vertices.)

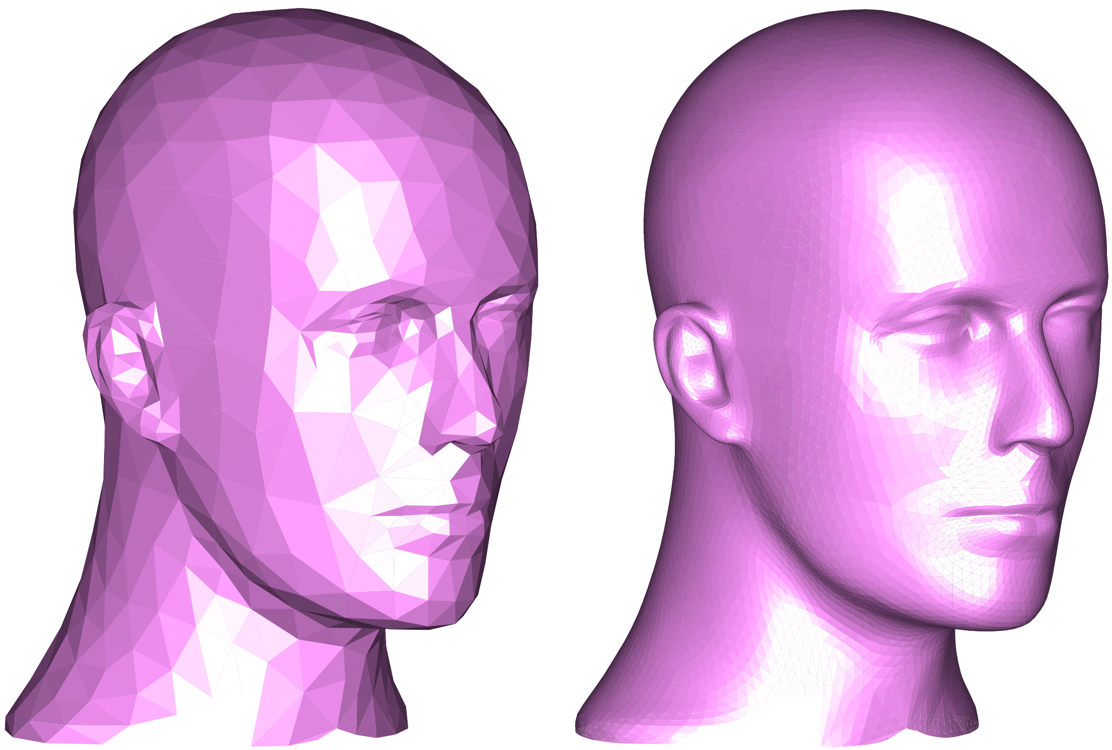

Figure 3. Application of two steps of the 2-Point Hermite subdivision scheme to the Mannequin Head dataset.

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