Continuity

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 1.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Know what it means for a function to be continuous at a specific value and on an interval.
- Find values where a function is not continuous; specifically, you should be able to do this for polynomials, rational functions, exponential and logarithmic functions, and other elementary functions.
- Determine the values for which a piecewise function is discontinuous, if any such values exist.
- Use the Intermediate Value Theorem to show the existence of a solution to an equation.

PRACTICE PROBLEMS:

Use the graph of $f(x)$, shown below, to answer questions 1-3

1. For which values of $x$ is $f(x)$ discontinuous?

   $f(x)$ is discontinuous when $x = 0$, $x = 3$, and $x = 6$. 

2. At each point of discontinuity, explain why $f(x)$ is discontinuous.

- At $x = 0$, $f(x)$ is discontinuous because $\lim_{x \to 0} f(x)$ DNE.
- At $x = 3$, $f(x)$ is discontinuous because $\lim_{x \to 3} f(x) \neq f(3)$.
- At $x = 6$, $f(x)$ is discontinuous because $f(6)$ is undefined.

3. Determine whether $f(x)$ is continuous on the given interval. If not, explain why.

   (a) $[-8, -4]$  
      Yes
   
   (b) $[-8, 0]$  
      No because $\lim_{x \to 0^-} f(x) \neq f(0)$
   
   (c) $[-8, 0)$  
      Yes
   
   (d) $[-2, 1]$  
      No because $\lim_{x \to 0} f(x)$ DNE
   
   (e) $(3, 6)$  
      Yes
   
   (f) $[3, 6)$  
      No because $\lim_{x \to 3^+} f(x) \neq f(3)$
   
   (g) $(6, 9]$  
      Yes
   
   (h) $[6, 9]$  
      No because $f(6)$ is undefined

4. For each of the following, sketch the graph of a function, $y = f(x)$, which satisfies the given characteristic. (There are many possible answers for each)

   (a) $f(x)$ is continuous everywhere except at $x = 1$.  
      Any graph for which either $f(1)$ is undefined or $\lim_{x \to 1} f(x)$ DNE or $\lim_{x \to 1} f(x) \neq f(1)$
   
   (b) $f(x)$ is continuous everywhere except at $x = -2$ where the $\lim_{x \to -2} f(x) = \lim_{x \to -2^+} f(x)$.  
      Any graph for which either $f(-2)$ is undefined or $\lim_{x \to -2} f(x) \neq f(-2)$
   
   (c) $f(x)$ is continuous everywhere except at $x = 0$, where $f(0) = 2$.  
      Any graph for which $\lim_{x \to 0} f(x)$ DNE or $\lim_{x \to 0} f(x) \neq 2$
5. Sketch the graph of a function which satisfies the following criteria:

- The domain of \( f(x) \) is \([1, 3]\)
- \( f(x) \) is continuous on \([1, 2]\) and \((2, 3]\).
- \( f(x) \) is not continuous on \([1, 3]\)

For problems 6-15, determine the value(s) of \( x \) where the given function has a point of discontinuity, if any such values exist.

6. \( f(x) = |x| \)
   \( f(x) \) is always continuous

7. \( f(x) = x^2 - x - 5 \)
   \( f(x) \) is always continuous

8. \( f(x) = \frac{x}{x - 1} \)
   \( f(x) \) has a discontinuity when \( x = 1 \)

9. \( f(x) = \sqrt{x - 1} \)
   \( f(x) \) is always continuous

10. \( f(x) = \frac{x^2 + 3x - 10}{x - 7} \)
    \( f(x) \) has discontinuity when \( x = 7 \)
11. \( f(x) = \frac{x^2 - 4}{x - 2} \)

\( f(x) \) has a discontinuity when \( x = 2 \)

12. \( f(x) = \frac{1}{x^2 - 2} + \frac{x^3 - 1}{2x^2 - 1} \)

\( f(x) \) has a discontinuity when \( x = \sqrt{2}, x = -\sqrt{2}, x = \frac{\sqrt{2}}{2}, \) and \( x = -\frac{\sqrt{2}}{2} \)

13. \( f(x) = \begin{cases} 
  x^2 - 1, & \text{if } x < 2 \\
  \frac{3}{x - 1}, & \text{if } x \geq 2
\end{cases} \)

\( f(x) \) is always continuous

14. \( f(x) = \begin{cases} 
  5 + \frac{1}{x}, & \text{if } x < -1 \\
  3x^2 + 2x + 3, & \text{if } x > -1
\end{cases} \)

\( f(x) \) has a discontinuity when \( x = -1 \)

15. \( f(x) = \begin{cases} 
  x^2 - 3x + 4, & \text{if } x \leq 1 \\
  x^4 - 4x^3 - 2x^2 + 6, & \text{if } x > 1
\end{cases} \)

\( f(x) \) has discontinuity when \( x = 1 \)

16. Find the value(s) of \( k \) such that \( f(x) \) is continuous everywhere:

\[ f(x) = \begin{cases} 
  x^2 - 7, & \text{if } x \leq 2 \\
  4x^3 - 3kx + 2, & \text{if } x > 2
\end{cases} \]

\[ k = \frac{37}{6} \]

17. Find the value(s) of \( k \) and \( m \) such that \( f(x) \) is continuous everywhere:

\[ f(x) = \begin{cases} 
  2x + 8m, & \text{if } x \leq -2 \\
  mx + k, & \text{if } -2 < x \leq 2 \\
  -3x^2 + 8x - 2k, & \text{if } x > 2
\end{cases} \]

\[ m = \frac{1}{2} \text{ and } k = 1 \]
18. **Multiple Choice:** Where is \( f(x) = \frac{\sqrt{x - 2}}{x^2 - x} \) continuous?

(a) \( x \neq 0 \) and \( x \neq 1 \)
(b) \( x \leq 2 \) where \( x \neq 0 \) and \( x \neq 1 \)
(c) \( x \leq 2 \)
(d) \( x \geq 2 \)
(e) \( |x| > 2 \)

19. Consider the following definitions:

- **Definition:** A function \( f(x) \) has a [removable discontinuity](#) at \( x = a \) if \( \lim_{x \to a} f(x) \) exists but \( f(x) \) is not continuous at \( x = a \). This could be because \( f(a) \) is undefined or because \( \lim_{x \to a} f(x) \neq f(a) \).

- **Definition:** A function \( f(x) \) has a [jump discontinuity](#) at \( x = a \) if \( \lim_{x \to a^-} f(x) \) exists and \( \lim_{x \to a^+} f(x) \) exists, but \( \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \)

For each of the following, determine the value(s) of \( x \) where the given function has a point of discontinuity. Classify each discontinuity as a removable discontinuity, a jump discontinuity, or neither.

(a) \( f(x) = \frac{x^2 - 4}{x - 2} \)

\( f(x) \) has a removable discontinuity when \( x = 2 \)

(b) \( f(x) = \frac{x - 1}{x - 4} \)

\( f(x) \) has a discontinuity when \( x = 4 \); it is neither a removable discontinuity nor a jump discontinuity.

(c) \( f(x) = \begin{cases} x^2 - 3x + 4, & \text{if } x \leq 1 \\ x^4 - 4x^3 - 2x^2 + 6, & \text{if } x > 1 \end{cases} \)

\( f(x) \) has jump discontinuity when \( x = 1 \)

(d) \( f(x) = \frac{x - 1}{x^2 - 4x + 3} \)

\( f(x) \) has a removable discontinuity when \( x = 1 \). \( f(x) \) has another discontinuity when \( x = 3 \); it is neither a removable discontinuity nor a jump discontinuity.
20. **Multiple Choice:** Consider the function:

\[ f(x) = \begin{cases} 
  x^2 & \text{if } x < -2 \\
  4 & \text{if } -2 < x \leq 1 \\
  6 - x & \text{if } x > 1 
\end{cases} \]

Which of the following statements is true about \( f(x) \)?

(a) \( f(x) \) is continuous everywhere.
(b) If \( f(-2) \) were defined to be 4, then \( f(x) \) would be continuous everywhere.
(c) The only discontinuity of \( f(x) \) occurs when \( x = -2 \).
(d) The only discontinuity of \( f(x) \) occurs when \( x = 1 \).
(e) The only discontinuities of \( f(x) \) occur when \( x = -2 \) and \( x = 1 \).

21. Show that the equation \( x^3 - x^2 + 3x - 1 = 1 \) has at least one solution in \((0, 1)\).

Let \( f(x) = x^3 - x^2 + 3x - 2 \). It suffices to show that there exists a \( c \) in \((0, 1)\) such that \( f(c) = 0 \). Since \( f(x) \) is a polynomial, it is continuous everywhere on \((\infty, \infty)\). Specifically, it is continuous on \([0, 1]\). Since \( f(0) = -2 < 0 \) and \( f(1) = 1 > 0 \), the Intermediate Value Theorem states that there exists some \( c \in (0, 1) \), \( f(c) = 0 \). The result follows.

22. Show that \( f(x) = x^3 - 9x + 5 \) has at least one \( x \)-intercept in \((1, 10)\).

We need to show that there exists at least one solution to \( f(x) = 0 \). Since \( f(x) \) is a polynomial, it is continuous on \([1, 10]\). Notice that \( f(1) = -3 < 0 \) and \( f(10) = 915 > 0 \). Thus, the Intermediate Value Theorem states that there must be a \( c \in (1, 10) \) with \( f(c) = 0 \).

23. Use the intermediate value theorem to show that \( x^3 - 2x^2 - 2x + 1 = 0 \) has at least **TWO** solutions in \([0, 5]\).

We will apply the IVT twice – first on \([0, 1]\) and then on \([1, 5]\). Let \( f(x) = x^3 - 2x^2 - 2x + 1 \). Since \( f(x) \) is a polynomial, it is continuous on \((-\infty, \infty)\). As a result, it is continuous on \([0, 1]\) and \([1, 5]\). Notice that \( f(0) = 1 > 0 \) and \( f(1) = -2 < 0 \). So, the IVT implies that there exists a \( c \) in \((0, 1)\) such that \( f(c) = 0 \). Similarly, notice that \( f(1) = -2 < 0 \) and \( f(5) = 66 > 0 \). So, the IVT implies that there exists a \( d \) in \((1, 5)\) such that \( f(d) = 0 \).