Chapter 2.5 Practice Problems

EXPECTED SKILLS:

• Know the derivatives of the 6 elementary trigonometric functions.
• Be able to use these derivatives in the context of word problems.

PRACTICE PROBLEMS:

1. Fill in the given table:

<table>
<thead>
<tr>
<th>f(x)</th>
<th>f'(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin x</td>
<td></td>
</tr>
<tr>
<td>cos x</td>
<td></td>
</tr>
<tr>
<td>tan x</td>
<td></td>
</tr>
<tr>
<td>cot x</td>
<td></td>
</tr>
<tr>
<td>sec x</td>
<td></td>
</tr>
<tr>
<td>csc x</td>
<td></td>
</tr>
</tbody>
</table>

2. Use the definition of the derivative to show that \( \frac{d}{dx}(\cos x) = -\sin x \)

   Hint: \( \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \)

\[
\begin{align*}
\frac{d}{dx}(\cos x) &= \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h \to 0} \left( \frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right) \\
&= \lim_{h \to 0} \left( \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) \\
&= (\cos x)(0) - (\sin x)(1) \\
&= -\sin x
\end{align*}
\]
3. Use the quotient rule to show that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

$$
\frac{d}{dx}(\cot x) = \frac{d}{dx} \left( \frac{\cos x}{\sin x} \right) \\
= \frac{\sin x (\cos x) - \cos x (\sin x)}{\sin^2 x} \\
= \frac{-\sin^2 x}{\sin^2 x} \\
= -1 \\
= -\csc^2 x
$$

4. Use the quotient rule to show that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

$$
\frac{d}{dx}(\csc x) = \frac{d}{dx} \left( \frac{1}{\sin x} \right) \\
= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} \\
= -\frac{\cos x}{\sin^2 x} \\
= -\frac{1}{\sin x \sin x} \\
= -\frac{1}{\sin x \cos x} \\
= -\csc x \cot x
$$

5. Evaluate $\lim_{h \to 0} \frac{\tan \left( \frac{\pi}{3} + h \right) - \tan \left( \frac{\pi}{3} \right)}{h}$ by interpreting the limit as the derivative of a function at a particular point.

$$
\lim_{h \to 0} \frac{\tan \left( \frac{\pi}{3} + h \right) - \tan \left( \frac{\pi}{3} \right)}{h} = \left. \frac{d}{dx}(\tan x) \right|_{x=\frac{\pi}{3}} = \sec^2 \left( \frac{\pi}{3} \right) = 4
$$

For problems 6-14, differentiate

6. $f(x) = 2 \cos x + 4 \sin x$

$$
-2 \sin x + 4 \cos x
$$

7. $f(x) = 5 \cos x + \cot x$

$$
-5 \sin x - \csc^2 x
$$
8. \( g(x) = 4 \csc x + 2 \sec x \)
\[
-4 \csc (x) \cot (x) + 2 \sec (x) \tan (x)
\]
9. \( f(x) = \sin x \cos x \)
\[
\cos^2 x - \sin^2 x
\]
10. \( f(x) = \frac{\sin^2 x}{\cos x} \)
\[
2 \sin x + \sin x \tan^2 x
\]
11. \( f(x) = x^3 \sin x \)
\[
3x^2 \sin x + x^3 \cos x
\]
12. \( f(x) = \sec^2 x + \tan^2 x \)
\[
4 \sec^2 (x) \tan (x)
\]
13. \( f(x) = \frac{x + \sec x}{1 + \cos x} \)
\[
\frac{1 + 2 \tan x + \cos x + \sec (x) \tan (x) + x \sin x}{(1 + \cos x)^2}
\]

For problems 14-17, compute \( \frac{d^2y}{dx^2} \)

14. \( f(x) = \tan x \)
\[
2 \sec^2 x \tan x
\]
15. \( f(x) = \sin x \)
\[
-\sin x
\]
16. \( f(x) = \cos^2 x \)
\[
2 \sin^2 x - 2 \cos^2 x
\]
17. \( f(x) = \sin^2 x + \cos^2 x \)
\[
0
\]
For problems 18-19, find all values of $x$ in the interval $[0, 2\pi]$ where the graph of the given function has horizontal tangent lines.

18. $f(x) = \sin x \cos x$

\[
\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}; \text{http://www.youtube.com/watch?v=sfsjC-6qMMs}
\]

19. $g(x) = \csc x$

\[
\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}
\]

20. Compute an equation of the line which is tangent to the graph of $f(x) = \frac{\cos x}{x}$ at the point where $x = \pi$.

\[
y = \frac{1}{\pi^2}x - \frac{2}{\pi}
\]

21. Consider the graphs of $f(x) = \sqrt{2}\cos(x)$ and $g(x) = \sqrt{2}\sin(x)$ shown below on the interval $[0, \frac{\pi}{2}]$.

Show that the graphs of $f(x)$ and $g(x)$ intersect at a right angle when $x = \frac{\pi}{4}$. (Hint: Show that the tangent lines to $f$ and $g$ at $x = \frac{\pi}{4}$ are perpendicular to each other.)

\[
f'(\frac{\pi}{4}) = -1 \text{ and } g'(\frac{\pi}{4}) = 1. \text{ So, the tangent lines to } f \text{ and } g \text{ at } x = \frac{\pi}{4} \text{ are perpendicular to one another since the product of their slopes is } -1.
\]
22. A 15 foot ladder leans against a vertical wall at an angle of $\theta$ with the horizontal, as shown in the figure below. The top of the ladder is $h$ feet above the ground. If the ladder is pushed towards the wall, find the rate at which $h$ changes with respect to $\theta$ at the instant when $\theta = 30^\circ$. Express your answer in feet/degree.

\[
\frac{dh}{d\theta} = \frac{15\sqrt{3}}{2} \text{ ft/radian} = \frac{\pi \sqrt{3}}{24} \text{ ft/degree}
\]

23. Use the Intermediate Value Theorem to show that there is at least one point in the interval $(0, 1)$ where the graph of $f(x) = \sin x - \frac{1}{3}x^3$ will have a horizontal tangent line.

$f'(x) = \cos x - x^2$. Firstly, notice that $f'(x)$ is continuous for all $x$; therefore, it is continuous for all $x$ in $[0, 1]$. Secondly, notice that $f'(0) = 1 > 0$ and $f'(1) = \cos (1) - 1 < 0$. Thus, the Intermediate Value Theorem states there is at least one $x_0$ in the interval $(0, 1)$ with $f'(x_0) = 0$. In other words, there is at least one $x_0$ in $(0, 1)$ where $f(x)$ will have a horizontal tangent line.

24. **Multiple Choice:** At how many points on the interval $[-\pi, \pi]$ is the tangent line to the graph of $y = 2x + \sin x$ parallel to the secant line which passes through the graph endpoints of the interval?

(a) 0  
(b) 1  
(c) 2  
(d) 3  
(e) None of these

\[C\]