Analysis of Functions I: 
Increasing, Decreasing & Concavity

SUGGESTED REFERENCE MATERIAL:
As you work through the problems listed below, you should reference Chapters 4.1 & 4.2 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Understand how the signs of the first and second derivatives of a function are related to the behavior of the function.

- Know how to use the first and second derivatives of a function to find intervals on which the function is increasing, decreasing, concave up, and concave down.

- Be able to find the critical points of a function, and apply the First Derivative Test and Second Derivative Test (when appropriate) to determine if the critical points are relative maxima, relative minima, or neither

- Know how to find the locations of inflection points.

PRACTICE PROBLEMS:

1. Consider the graph of $y = f(x)$, shown below.
(a) Determine the interval(s) where \( f(x) \) is increasing.

\[ (b, d) \cup (f, \infty) \]

(b) Determine the interval(s) where \( f(x) \) is decreasing.

\[ (-\infty, b) \cup (d, f) \]

(c) Determine the interval(s) where \( f(x) \) is concave up.

\[ (-\infty, c) \cup (e, \infty) \]

(d) Determine the interval(s) where \( f(x) \) is concave down.

\[ (c, e) \]

(e) Determine the value(s) of \( x \) where \( f(x) \) has relative (local) extrema. Classify each as the location of a relative maximum or a relative minimum.

Relative max when \( x = d \); Relative minima when \( x = b \) and \( x = f \)

(f) Determine the value(s) of \( x \) where \( f(x) \) has an inflection point.

Point of Inflection when \( x = c \) and \( x = e \)

2. The graph of the derivative of \( y = f(x) \) is shown below.

![Graph of derivative](image)

(a) Determine the interval(s) where \( f(x) \) is increasing.

\[ (-\infty, a) \cup (g, \infty) \]

(b) Determine the interval(s) where \( f(x) \) is decreasing.

\[ (a, d) \cup (d, g) \]

(c) Determine the interval(s) where \( f(x) \) is concave up.

\[ (b, d) \cup (f, \infty) \]
(d) Determine the interval(s) where $f(x)$ is concave down.

\[ (-\infty, b) \cup (d, f) \]

(e) Determine the value(s) of $x$ where $f(x)$ has relative (local) extrema. Classify each as the location of a relative maximum or a relative minimum.

Relative maximum when $x = a$; Relative minimum when $x = g$; Neither a relative max nor a relative min at the critical point of $x = d$.

(f) Determine the value(s) of $x$ where $f(x)$ has an inflection point.

Points of inflection when $x = b$, $x = d$ and $x = f$

3. Sketch the graph of a continuous function, $y = f(x)$, which is decreasing on $(-\infty, \infty)$, has an inflection point at $x = 1$, and is concave down on $(1, \infty)$.

![Graph of a continuous function decreasing on $(-\infty, \infty)$ with an inflection point at $x = 1$ and concave down on $(1, \infty)$]

4. Sketch the graph of a continuous function, $y = f(x)$, which is decreasing on $(-\infty, 1)$, has a relative minimum at $x = 1$, and does not have any inflection points.

![Graph of a continuous function decreasing on $(-\infty, 1)$ with a relative minimum at $x = 1$ and no inflection points]

or

![Another graph of a continuous function decreasing on $(-\infty, 1)$ with a relative minimum at $x = 1$ and no inflection points]

5. Sketch the graph of a continuous function $y = f(x)$ which satisfies all of the following conditions:

- Domain of $f(x)$ is $(-\infty, \infty)$
- $f(-1) = -2$, $f(0) = f(7) = 3$, and $f(5) = 9$
- $f'(x) < 0$ on $(-\infty, -1) \cup (5, 7)$ and $f'(x) > 0$ on $(-1, 0) \cup (0, 5) \cup (7, \infty)$
- $f''(x) < 0$ on $(0, 7) \cup (7, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$
6. Consider the function that you sketched in question 5. At which value(s) of \( x \) must \( f'(x) = 0 \)? At which value(s) of \( x \) must \( f'(x) \) fail to exist?

\[
f'(x) = 0 \text{ when } x = -1 \text{ and } x = 5; \quad f'(x) \text{ DNE when } x = 7
\]

For problems 7-15, calculate each of the following:

(a) The intervals on which \( f(x) \) is increasing
(b) The intervals on which \( f(x) \) is decreasing
(c) The intervals on which \( f(x) \) is concave up
(d) The intervals on which \( f(x) \) is concave down
(e) All points of inflection. Express each as an ordered pair \((x, y)\)

7. \( f(x) = x^3 - 2x + 3 \)

| a. \((-\infty, -\sqrt[3]{\frac{2}{3}}) \cup \left(\sqrt[3]{\frac{2}{3}}, \infty\right)\) | b. \((-\sqrt[3]{\frac{2}{3}}, \sqrt[3]{\frac{2}{3}})\) | c. \((0, \infty)\) | d. \((-\infty, 0)\) | e. \((0, 3)\) |

8. \( f(x) = \frac{x}{x - 2} \)

| a. none | b. \((-\infty, 2) \cup (2, \infty)\) | c. \((2, \infty)\) | d. \((-\infty, 2)\) | e. none |

9. \( f(x) = \sin x \) on \([0, 2\pi]\)

| a. \([0, \frac{\pi}{2}) \cup \left(\frac{3\pi}{2}, 2\pi\right)\) | b. \(\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\) | c. \((\pi, 2\pi)\) | d. \((0, \pi)\) | e. \((\pi, 0)\) |

10. \( f(x) = (4x - 1)^4 \)

| a. \((\frac{1}{4}, \infty)\) | b. \((-\infty, \frac{1}{4})\) | c. \((-\infty, \frac{1}{4}) \cup \left(\frac{1}{4}, \infty\right)\) | d. none | e. none |

11. \( f(x) = xe^x \)

| a. \((-1, \infty)\) | b. \((-\infty, -1)\) | c. \((-2, \infty)\) | d. \((-\infty, -2)\) | e. \((-2, -\frac{2}{e^2})\) |
12. \( f(x) = \arctan(2x) \)
   
   a. \((-\infty, \infty)\); b. none; c. \((-\infty, 0)\); d. \((0, \infty)\); e. \((0, 0)\)

13. \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \)
   
   a. \((-\infty, 0)\); b. \((0, \infty)\); c. \((-\infty, -1) \cup (1, \infty)\); d. \((-1, 1)\); e. \((-1, \frac{1}{\sqrt{2\pi e^{1/2}}} ) \) and \((1, \frac{1}{\sqrt{2\pi e^{1/2}}} ) \)

14. \( f(x) = \frac{\ln x}{x} \)
   
   a. \((0, e)\); b. \((e, \infty)\); c. \((e^{3/2}, \infty)\); d. \((0, e^{3/2})\); e. \(\left(e^{3/2}, \frac{3}{2e^{3/2}}\right)\)

15. \( f(x) = 2x + 3x^{2/3} \)
   
   a. \((-\infty, -1) \cup (0, \infty)\); b. \((-1, 0)\); c. none; d. \((-\infty, 0) \cup (0, \infty)\); e. none

For problems 16-20, compute the critical points of the given function. Then use the First Derivative Test to determine all relative (local) extrema. Express each extremum as an ordered pair \((x, y)\).

16. \( f(x) = x^2 - 16 \)
   
   Relative min at \((0, -16)\)

17. \( f(x) = (2x + 3)^3 \)
   
   Critical Point at \(-\frac{3}{2}\), No relative extrema

18. \( f(x) = \frac{3x}{x^2 + 1} \)
   
   Relative max at \(\left(1, \frac{3}{2}\right)\); Relative min at \((-1, -\frac{3}{2})\)

19. \( f(x) = e^x - x \)
   
   Relative min at \((0, 1)\)

20. \( f(x) = x^3 - x^5 \)
   
   Relative maximum at \(\left(\sqrt{\frac{3}{5}}, \frac{2}{5}, \left(\frac{3}{5}\right)^{3/2}\right)\)
   
   Relative minimum at \(\left(-\sqrt{\frac{3}{5}}, \frac{2}{5}, \left(\frac{3}{5}\right)^{3/2}\right)\)
   
   Critical point at \((0, 0)\), which is neither a relative max nor a relative min
For problems 21-22, use the Second Derivative Test to determine the relative (local) extrema. Express each as an ordered pair \((x, y)\).

21. \(f(x) = \sin (3x)\) on \([0, \pi]\)
   
   | Relative maxima at \(\left(\frac{\pi}{6}, 1\right)\) and \(\left(\frac{5\pi}{6}, 1\right)\); Relative minimum at \(\left(\frac{\pi}{2}, -1\right)\) |

22. \(f(x) = \sec (3x)\) on \([0, \pi]\)
   
   | Relative minima at \((0, 1)\) and \(\left(\frac{2\pi}{3}, 1\right)\); Relative maxima at \(\left(\frac{\pi}{3}, -1\right)\) and \((\pi, -1)\) |

For problems 23-27, determine the critical points. Classify each as a relative extremum, relative minimum, or neither. Express all relative extrema as ordered pairs \((x, y)\).

23. \(f(x) = \sin^2 x\) on \([0, 2\pi]\)
   
   | Relative minima at \((0, 0)\), \((\pi, 0)\), and \((2\pi, 0)\); Relative maxima at \(\left(\frac{\pi}{2}, 1\right)\) and \(\left(\frac{3\pi}{2}, 1\right)\) |

24. \(f(x) = \frac{x^3}{3} + x^2 + x + 3\)
   
   No relative extrema

25. \(f(x) = xe^x\)
   
   | Relative minimum at \(\left(-1, -\frac{1}{e}\right)\) |

26. \(f(x) = 2x + 3x^{2/3}\)
   
   | Relative maximum at \((-1, 1)\); Relative minimum at \((0, 0)\) |

27. \(f(x) = \frac{\ln x}{x}\)
   
   | Relative Maximum at \(\left(e, \frac{1}{e}\right)\) |

HINT: For problems 25-27, it may be helpful to use your work from earlier in the assignment.
28. Suppose $f(x)$ and $g(x)$ are twice differentiable functions and that $x_0$ is a stationary point of $g(x)$. Use the second derivative test to prove that if $x_0$ is the location of a relative maximum of $g(x)$, then $x_0$ is also the location of a relative maximum of $f(g(x))$.

Since $x_0$ is a stationary point of $g(x)$, we have $g'(x_0) = 0$. And, since $x_0$ is the location of a relative maximum of $g(x)$, we have $g''(x_0) < 0$. To show that $x_0$ is the location of a relative maximum of $f(g(x))$, we will show both of the following:

• \[ \frac{d}{dx} (f(g(x))) \bigg|_{x=x_0} = 0 \]

• \[ \frac{d^2}{dx^2} (f(g(x))) \bigg|_{x=x_0} < 0 \]

Firstly, \[ \frac{d}{dx} (f(g(x))) = f'(g(x))g'(x) \implies \frac{d}{dx} (f(g(x))) \bigg|_{x=x_0} = f'(g(x_0))g'(x_0) = 0 \] since $x_0$ is a stationary point of $g(x)$. Thus, $x_0$ is also stationary point of $f(g(x))$.

Secondly, by the product rule we have:

\[ \frac{d^2}{dx^2} (f(g(x))) = f''(g(x))[g'(x)]^2 + f'(g(x))g''(x) \]

Evaluating this at $x_0$ gives

\[ \frac{d^2}{dx^2} (f(g(x))) \bigg|_{x=x_0} = f''(g(x_0))[g'(x_0)]^2 + f'(g(x_0))g''(x_0) \]

But, since $x_0$ is a stationary point of $g(x)$, this simplifies to:

\[ \frac{d^2}{dx^2} (f(g(x))) \bigg|_{x=x_0} = f'(g(x_0))g''(x_0) \]

Since $x_0$ is the location of a relative maximum of $g(x)$, $g''(x_0) < 0$. And, since $f(x)$ is strictly increasing, $f'(x) > 0$ for all $x$ in its domain. Thus, \[ \frac{d^2}{dx^2} (f(g(x))) \bigg|_{x=x_0} < 0 \] and $x_0$ is the location of a relative maximum of $f(g(x))$.

29. Use exercise 28 to determine the value(s) of $x$ for which \( y = \sqrt{9 + 2x - 4x^2} \) achieves a local maximum.

The function will have a relative maximum when $x = \frac{1}{4}$