Length of a Plane Curve (Arc Length)

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 6.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

• Be able to find the arc length of a smooth curve in the plane described as a function of $x$ or as a function of $y$.

PRACTICE PROBLEMS:

For problems 1-3, compute the exact arc length of the curve over the given interval.

1. $y = 4x^{\frac{3}{2}} - 1$ from $x = \frac{1}{12}$ to $x = \frac{2}{9}$

\[
\frac{19}{54}
\]

2. $y = \frac{x^2}{2} - \frac{\ln(x)}{4}$ for $2 \leq x \leq 4$

\[
6 + \frac{1}{4} \ln 2; \text{ Detailed Solution: Here}
\]

3. $y = \frac{2}{3}(x^2 - 1)^{3/2}$ for $1 \leq x \leq 3$

\[
\frac{46}{3}
\]

4. Consider the curve defined by $y = \sqrt{4 - x^2}$ for $0 \leq x \leq 2$.

(a) Compute the arc length on the interval $[0, t]$ for $0 \leq t < 2$. (Your arc length will depend on $t$.)

\[
2 \sin^{-1}\left(\frac{t}{2}\right)
\]

(b) Use your answer from part (a) to compute the arc length on the interval $[0, 2]$. (Hint: You will need to introduce a limit.)

$\pi$
(c) Confirm your answer from part (b) by using geometry.

On the interval [0, 2], the curve is \(\frac{1}{4}\) of a circle with a radius of 2. So, the length should be \(\frac{1}{4}\) of the circumference; that is, 
\[
\text{Length} = \frac{1}{4} \cdot 2\pi r \bigg|_{r=2} = \frac{1}{4} \cdot 2\pi(2) = \pi.
\]

5. Consider \(F(x) = \int_1^x \sqrt{t^2 - 1} \, dt\). Compute the arc length on \([1, 3]\)

4; Detailed Solution: [Here]

6. Consider the curve defined by \(f(x) = \ln x\) on \([1, e^3]\)

(a) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to \(x\).
\[
L = \int_1^{e^3} \sqrt{1 + \frac{1}{x^2}} \, dx
\]

(b) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to \(y\).
\[
L = \int_0^{e^3} \sqrt{1 + e^{2y}} \, dy
\]

7. Consider the curve defined by \(f(x) = \tan x\) on \([-\frac{\pi}{3}, \frac{\pi}{4}]\)

(a) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to \(x\).
\[
L = \int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \sqrt{1 + \sec^4 x} \, dx
\]

(b) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to \(y\).
\[
L = \int_{-\sqrt{3}}^1 \sqrt{1 + \frac{1}{(1 + y^2)^2}} \, dy
\]

8. Consider the curve defined by \(y = \sin x\) for \(0 \leq x \leq \pi\).

(a) Set up but do not evaluate an integral which represents the length of the curve.
\[
\int_0^\pi \sqrt{1 + \cos^2 x} \, dx
\]
(b) Estimate the value of your integral from part (a) by using a Midpoint Approximation with three rectangles of equal width.

Below is the graph of \( y = \sqrt{1 + \cos^2 x} \) on the interval \([0, \pi]\) along with three rectangles of equal width whose heights were determined by the function value at the midpoint of each resulting subinterval.

Using these rectangles, \( \int_{0}^{\pi} \sqrt{1 + \cos^2 x} \, dx \approx \frac{\pi}{3} \left( 1 + \sqrt{7} \right) \)

9. Recall the definitions of Hyperbolic Sine & Hyperbolic Cosine from Math 121:

\[
\begin{align*}
\text{Hyperbolic Sine} & \quad \text{Hyperbolic Cosine} \\
\sinh x &= \frac{e^x - e^{-x}}{2} & \cosh x &= \frac{e^x + e^{-x}}{2}
\end{align*}
\]

The sketches of \( y = \cosh x \) and \( y = \sinh x \) are shown below. The dashed curves are called “Curvilinear Asymptotes,” which describe the end behavior of the functions.
(a) Show that $\cosh^2 x - \sinh^2 x = 1$

\[
\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x) \\
= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) \\
= (e^x)(e^{-x}) \\
= 1
\]

(b) Verify that $f(x) = \sinh x$ is an odd function. (Hint: Recall an odd function satisfies the identity $f(-x) = -f(x).$)

To verify that a function is odd, we check that $f(-x) = -f(x).$ We compute by appealing to the definition of $\sinh x$ from above.

\[
\sinh (-x) = \frac{e^{-x} - e^{-(x)}}{2} \\
= \frac{e^{-x} - e^x}{2} \\
= -\left(\frac{e^x - e^{-x}}{2}\right) \\
= -\sinh x
\]

Thus, $f(x) = \sinh x$ is odd.
(c) Show that \( \frac{d}{dx} (\sinh x) = \cosh x \) and deduce that \( \int \cosh x \, dx = \sinh x + C \).

\[
\frac{d}{dx} (\sinh x) = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) \\
= \frac{d}{dx} \left( \frac{1}{2} e^x - \frac{1}{2} e^{-x} \right) \\
= \frac{1}{2} e^x + \frac{1}{2} e^{-x} \\
= \frac{e^x + e^{-x}}{2} \\
= \cosh x
\]

Thus, as a result, \( \int \cosh x \, dx = \sinh x + C \). (We could have also verified this integration formula by integrating the given definition of \( \cosh x \)).

(d) Show that \( \frac{d}{dx} (\cosh x) = \sinh x \) and deduce that \( \int \sinh x \, dx = \cosh x + C \).

\[
\frac{d}{dx} (\cosh x) = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right) \\
= \frac{d}{dx} \left( \frac{1}{2} e^x + \frac{1}{2} e^{-x} \right) \\
= \frac{1}{2} e^x - \frac{1}{2} e^{-x} \\
= \frac{e^x - e^{-x}}{2} \\
= \sinh x
\]

Thus, as a result, \( \int \sinh x \, dx = \cosh x + C \). (We could have also verified this integration formula by integrating the given definition of \( \sinh x \)).

(e) A telephone wire which is supported only by two telephone poles will sag under its own weight and form the shape of a **catenary** as shown below.
Consider a telephone wire that is supported by two poles (one at $x = b$ and the other at $x = -b$), as in the diagram below.

The shape of the sagging wire can be modeled by $y = a \cosh \left( \frac{x}{a} \right)$, where $a > 0$ and $-b \leq x \leq b$. What is the length of the wire?

$$A = 2a \sinh \left( \frac{b}{a} \right)$$