Improper Integrals

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.8 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

• Given an improper integral, which either has an infinite interval of integration or an infinite discontinuity, be able to evaluate it using a limit.

• Know how to determine if such an integral converges (and if so, what it converges to) or diverges.

PRACTICE PROBLEMS:

For problems 1-13, evaluate each improper integral or show that it diverges.

1. \[ \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx \]

2. \[ \int_{-\infty}^{3} \frac{3x}{x^2 + 1} \, dx \]

3. \[ \int_{1}^{\infty} e^{-x} \, dx \]

4. \[ \int_{1}^{\infty} xe^{-3x^2} \, dx \]

5. \[ \int_{0}^{4} \frac{1}{x^{2/3}} \, dx \]

\[ \frac{1}{6} e^{-3} \]

\[ 3^{\frac{2}{3}} \sqrt[3]{4} \]
6. \[ \int_{4}^{\infty} \frac{1}{(x-2)^3} \, dx \]
\[ \frac{1}{8} \]

7. \[ \int_{2}^{6} \frac{1}{\sqrt{x-2}} \, dx \]
\[ \frac{4}{3} \]

8. \[ \int_{0}^{2} \frac{2}{\sqrt{4-x^2}} \, dx \]
\[ \pi \]

9. \[ \int_{0}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx \] (Hint: Complete the square)
\[ \frac{\pi}{2} - \tan^{-1}(2) \]

10. \[ \int_{0}^{\pi} \frac{\sin x}{\sqrt{\cos x}} \, dx \]
\[ \frac{3}{2} \]

11. \[ \int_{0}^{\pi} \sqrt{\tan x \sec^2 x} \, dx \]
\[ \infty \]

12. \[ \int_{0}^{9} \frac{1}{\sqrt[3]{(x-1)^2}} \, dx \]
\[ 9 \]

13. \[ \int_{0}^{1} \frac{1}{x \ln x} \, dx \]
\[ -\infty \]

14. Find the value of the constant \( k \) so that \( \int_{-\infty}^{\infty} \frac{k}{1+x^2} \, dx = 1. \)
\[ \frac{1}{\pi} \]
15. Compute the exact area between the graph of \( y = \frac{4}{x^2 - 1} \) and the \( x \)-axis for \( x \geq 7 \).

\[ 2 \ln \left( \frac{4}{3} \right) \]

16. Consider Gabriel’s Horn (shown below) which is formed by revolving the curve \( y = \frac{1}{x} \) on \([1, \infty)\) around the \( x \)-axis.

Show that the volume within this horn is finite.

Note: It can be shown the surface area of this horn is infinite. Thus, it appears that the horn can be filled with a finite amount of paint; but, there is not enough to paint the inside of the surface for a coating of uniform thickness. This is called The **Paradox of Gabriel’s Horn**.

\[ V = \pi \text{ cubic units} \]

17. Determine the values of the constant \( p \) for which the following integral will converge and those for which it will diverge.

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx \]

(Hint: Consider two cases – when \( p = 1 \) and when \( p \neq 1 \).)

Converges to \( \frac{1}{p - 1} \) if \( p > 1 \); Diverges if \( p \leq 1 \); Detailed Solution: Here
18. Determine the values of the constant \( p \) for which the following integral will converge and those for which it will diverge.

\[
\int_0^1 \frac{1}{x^p} \, dx
\]

(Hint: Consider two cases - when \( p = 1 \) and when \( p \neq 1 \).)

Converges to \( \frac{1}{1-p} \) if \( p < 1 \); Diverges if \( p \geq 1 \)

19. Consider the Gamma Function: \( \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx \), which is defined for \( \alpha > 0 \).

(a) Compute \( \Gamma(1) \).

\[ 1 \]

(b) Compute \( \Gamma(2) \).

\[ 1 \]

(c) Compute \( \Gamma(3) \).

\[ 2 \]

20. If \( f(x) \) is a continuous for \( x \geq 0 \), the \textbf{Laplace Transform} of \( f(x) \) is given by:

\[
\mathcal{L} \{ f(x) \} (s) = \int_0^\infty f(x) e^{-sx} \, dx
\]

and the domain of \( \mathcal{L} \{ f(x) \} (s) \) is the set consisting of all numbers \( s \) for which the integral converges. Laplace Transforms are useful for solving differential equations

(a) Compute the Laplace Transform of \( f(x) = 1 \) and state its domain.

\[ \mathcal{L} \{ 1 \} (s) = \frac{1}{s} \text{ for } s > 0 \]

(b) Compute the Laplace Transform of \( f(x) = e^x \) and state its domain.

\[ \mathcal{L} \{ e^x \} (s) = \frac{1}{1-s} \text{ for } s > 1 \]

(c) Compute the Laplace Transform of \( f(x) = x \) and state its domain.

\[ \mathcal{L} \{ x \} (s) = \frac{1}{s^2} \text{ for } s > 0 \]
21. **Definition:** In probability theory, the **probability density function (pdf)** of a random variable $X$ is a function $f(x)$ from which we can compute the probability that $X$ lies in the interval $[a, b]$ as follows:

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx$$

And, in order for $f(x)$ to be a valid pdf, it must satisfy the following:

- $f(x) \geq 0$ for all values of $x$
- $\int_{-\infty}^{\infty} f(x) \, dx = 1$

(a) Verify that $f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ is a valid probability density function.

Notice that $f(x) > 0$ for all $x \geq 0$ because $e^{-2x} > 0$; and $f(x) = 0$ for $x < 0$. Thus, $f(x) \geq 0$ for all $x$. Also,

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_0^{\infty} 2e^{-2x} \, dx = 1$$

Thus, $f(x)$ is a valid pdf.

(b) Using the density function in part a, compute $P(0 \leq X \leq 1)$.

$$\int_0^1 2e^{-2x} \, dx = 1 - \frac{1}{e^2}$$

(c) **Definition:** The **cumulative distribution function (CDF)** for a continuous random variable $X$ is defined as:

$$F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x) \, dx$$

This CDF describes the accumulation of probability up to the real number $t$. Compute the CDF for the random variable $X$ which has the density function from part a.

The CDF is $F(t) = \int_{-\infty}^{t} f(x) \, dx = \int_0^t 2e^{-2x} \, dx = 1 - e^{-2t}$ for $t \geq 0$ and $0$ for $t < 0$. 

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