Polynomial Approximations of Functions

SUGGESTED REFERENCE MATERIAL:
As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Find and use the local linear and local quadratic approximations of a function \( f(x) \) at a specified \( x = x_0 \).
- Determine the Maclaurin polynomials of various degrees for a function \( f(x) \), and use sigma notation to write the \( n \)-th Maclaurin polynomial.
- Determine the Taylor polynomials of various degrees for a function \( f(x) \) at a specified \( x = x_0 \), and use sigma notation to write the \( n \)-th Taylor polynomial.

PRACTICE PROBLEMS:

1. Consider the function \( f(x) = \sqrt{x} \).

   (a) Find the local linear approximation \( p_1(x) \) and the local quadratic approximation \( p_2(x) \) to \( f(x) \) at \( x = 4 \).

   \[
   p_1(x) = 2 + \frac{1}{4}(x - 4) \\
   p_2(x) = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2
   \]

   Note that \( p_1(x) \) and \( p_2(x) \) are just the 1st and 2nd Taylor polynomials for \( f(x) \) about \( x = 4 \).

   (b) Approximate \( \sqrt{4.1} \) using your answers in part (a).

   \[
   p_1(4.1) = 2 + \frac{1}{4}(4.1 - 4) = \frac{81}{40} = 2.025 \\
   p_2(4.1) = 2 + \frac{1}{4}(4.1 - 4) - \frac{1}{64}(4.1 - 4)^2 = \frac{81}{40} - \frac{1}{6400} = 2.02484375
   \]

   Calculator: \( \sqrt{4.1} \approx 2.024845673 \).

For problems 2 – 4, use the appropriate local linear and local quadratic approximations to approximate the following values.

2. \( \sin 0.1 \)

   \[
   p_1(x) = p_2(x) = x, \text{ so } \sin 0.1 \approx 0.1 \\
   \text{Calculator: } \sin 0.1 \approx 0.09983341665
   \]
3. \( \sqrt[3]{28} \)

\[
p_1(x) = 3 + \frac{1}{27}(x - 27), \text{ so } \sqrt[3]{28} \approx p_1(28) = \frac{82}{27} = 3.037
\]

\[
p_2(x) = 3 + \frac{1}{27}(x - 27) - \frac{1}{2187}(x - 27)^2, \text{ so } \sqrt[3]{28} \approx p_2(28) = \frac{82}{27} - \frac{1}{2187} \approx 3.03657979
\]

Calculator: \( \sqrt[3]{28} \approx 3.036588972 \).

4. \( \tan 44^\circ \)

\[
p_1(x) = 1 + 2\left(x - \frac{\pi}{4}\right), \text{ so } \tan 44^\circ \approx p_1\left(\frac{\pi}{4} - \frac{\pi}{180}\right) = 1 - \frac{\pi}{90} \approx 0.965093415
\]

\[
p_2(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2, \text{ so } \tan 44^\circ \approx p_2\left(\frac{\pi}{4} - \frac{\pi}{180}\right) = 1 - \frac{\pi}{90} + \frac{2\pi^2}{(180)^2} \approx 0.9657026498
\]

Calculator: \( \tan 44^\circ \approx 0.9656887747 \).

5. Suppose that the values of \( f(x) \) and its first four derivatives at \( x = 0 \) are as follows:

\[
f(0) = 5 \quad f'(0) = -2 \quad f''(0) = 0 \quad f'''(0) = -1 \quad f^{(4)}(0) = 12
\]

Based on this information, list out as many Maclaurin polynomials for \( f(x) \) as possible.

\[
p_0(x) = 5
\]

\[
p_1(x) = p_2(x) = 5 - 2x
\]

\[
p_3(x) = 5 - 2x - \frac{1}{6}x^3
\]

\[
p_4(x) = 5 - 2x - \frac{1}{6}x^3 + \frac{1}{2}x^4
\]

6. Find the 4th Maclaurin polynomial \( p_4(x) \) for the function \( f(x) = 2x^4 - x^3 + 6 \).

\[
p_4(x) = 2x^4 - x^3 + 6. \text{ Why does it make sense that } p_4(x) = f(x)\?\]

For problem 7, find the Maclaurin polynomials \( p_0(x), p_1(x), p_2(x), p_3(x), \) and \( p_4(x) \). Then write the \( n \)-th Maclaurin polynomial \( p_n(x) \) using sigma notation.

7. \( f(x) = \ln(1 + x) \)
For problems 8 & 9, find the Taylor polynomials $p_0(x), p_1(x), p_2(x), p_3(x),$ and $p_4(x)$ about $x = x_0$. Then write the $n$-th Taylor polynomial $p_n(x)$ at $x = x_0$ using sigma notation.

8. $f(x) = \frac{1}{1-x}; \ x_0 = 2$

\[
p_0(x) = -1
\]
\[
p_1(x) = -1 + (x - 2)
\]
\[
p_2(x) = -1 + (x - 2) - (x - 2)^2
\]
\[
p_3(x) = -1 + (x - 2) - (x - 2)^2 + (x - 2)^3
\]
\[
p_4(x) = -1 + (x - 2) - (x - 2)^2 + (x - 2)^3 - (x - 2)^4
\]
\[
p_n(x) = \sum_{k=0}^{n} (-1)^{k+1} \frac{x^k}{k}
\]

9. $f(x) = e^{2x}; \ x_0 = \ln 3$
\begin{align*}
p_0(x) &= 9 \\
p_1(x) &= 9 + 18(x - \ln 3) \\
p_2(x) &= 9 + 18(x - \ln 3) + 18(x - \ln 3)^2 \\
p_3(x) &= 9 + 18(x - \ln 3) + 18(x - \ln 3)^2 + 12(x - \ln 3)^3 \\
p_4(x) &= 9 + 18(x - \ln 3) + 18(x - \ln 3)^2 + 12(x - \ln 3)^3 + 6(x - \ln 3)^4 \\
p_n(x) &= \sum_{k=0}^{n} \frac{2^k(9)}{k!} (x - \ln 3)^k
\end{align*}