Power Series

SUGGESTED REFERENCE MATERIAL:
As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Use sigma notation to write the Maclaurin series for a function $f(x)$.
- Use sigma notation to write the Taylor series for a function $f(x)$ about a specified $x = x_0$.
- Find the interval of convergence and the radius of convergence of a power series.
- Find the domain of a function that is expressed as a power series.

PRACTICE PROBLEMS:

For problems 1 & 2, use sigma notation to write the Maclaurin series for the given function.

1. $f(x) = \ln(1 + x)$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} ; \text{ Compare this to Polynomial Approximations of Functions } \#7.$$

2. $f(x) = x \cos x$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k)!}$$

For problems 3 & 4, use sigma notation to write the Taylor series for the given function about $x = x_0$.

3. $f(x) = e^{2x}; \ x_0 = \ln 3$

$$\sum_{k=0}^{\infty} \frac{2^k(9)}{k!} (x - \ln 3)^k ; \text{ Compare this to Polynomial Approximations of Functions } \#9.$$

4. $f(x) = \sin x; \ x_0 = \frac{\pi}{2}$

$$\sum_{k=0}^{\infty} (-1)^k \frac{(x - \frac{\pi}{2})^{2k}}{(2k)!}$$
For problems 5 – 13, find the interval of convergence and the radius of convergence $R$ for the power series.

5. $x + x^2 + x^3 + x^4 + \ldots$
   \[(-1, 1); R = 1. \text{ Note that this is the power series from Infinite Series #24.}\]

6. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$
   \[(-1, 1); R = 1. \text{ This is the Maclaurin series for } f(x) = \ln(1 + x). \text{ See problem #1.}\]

7. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$
   \[(-\infty, +\infty); R = +\infty. \text{ This is the Maclaurin series for } f(x) = e^x.\]

8. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$
   \[([-1, 1]; R = 1].\]

9. $\sum_{k=0}^{\infty} \frac{(-5)^k x^k}{\sqrt{k + 10}}$
   \[\left[\begin{array}{c}
   -\frac{1}{5}, \\
   \frac{1}{5}
   \end{array}\right]; R = \frac{1}{5}; \text{ Detailed Solution: Here}\]

10. $\sum_{k=0}^{\infty} \frac{(2k)! (2x + 1)^k}{(2k+1)!}$
    \[\left[\begin{array}{c}
    -\frac{1}{2}, \\
    -\frac{1}{2}
    \end{array}\right], \text{ or just } \left\{-\frac{1}{2}\right\}; R = 0. \text{ In other words, the series converges only when } x = -\frac{1}{2}.\]

11. $\sum_{k=0}^{\infty} \left(\frac{2}{7}\right)^k (x + 4)^{k+1}$
    \[\left[\begin{array}{c}
    -\frac{15}{2}, \\
    -\frac{1}{2}
    \end{array}\right]; R = \frac{7}{2}; \text{ Detailed Solution: Here}\]

12. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
    \[(-\infty, +\infty); R = +\infty. \text{ This is the Maclaurin series for } f(x) = \sin x.\]
For problems 14 – 16, a function is represented as a power series. Find the domain of the function.

14. \( f(x) = \sum_{k=0}^{\infty} \left[ (-1)^{k+1} (x - 2)^k \right] \)

The domain of \( f(x) \) is \( 1 < x < 3 \). This is the Taylor series for \( f(x) = \frac{1}{1-x} \) about \( x = 2 \). See Polynomial Approximations of Functions #8.

15. \( f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \)

The domain of \( f(x) \) is all real numbers. This is the Maclaurin series for \( f(x) = \cos x \).

16. \( f(x) = \sum_{k=0}^{\infty} \frac{e^{(k^2)}x^k}{k!} \)

The domain of \( f(x) \) is only \( x = 0 \); Detailed Solution: Here