First-Order Linear Equations (Integrating Factors)

SUGGESTED REFERENCE MATERIAL:
As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Be able to solve first-order linear equations by using the appropriate integrating factors.
- Be able to set up and solve application problems using integrating factors.

PRACTICE PROBLEMS:
For problems 1-6, use an integrating factor to solve the given differential equation. Express your answer as an explicit function of $x$.

1. $\frac{dy}{dx} - 4y = e^{5x}$
   
   $y = e^{5x} + Ce^{4x}$

2. $\frac{dy}{dx} + 3x^2y = x^2$
   
   $\frac{1}{3} + Ce^{-x^3}$

3. $y' = x - 2y$
   
   $Ce^{-2x} + \frac{1}{2}x - \frac{1}{4}$

4. $\frac{dy}{dx} - y = \sin(e^{-x})$
   
   $y = e^x \cos(e^{-x}) + Ce^x$

5. $y' + \frac{y}{x \ln x} = x$, for $x > 1$
   
   $y = \frac{1}{2}x^2 - \frac{x^2}{4 \ln x} + \frac{C}{\ln x}$

6. $\frac{dy}{dx} + y = \frac{1}{e^{2x} - 5e^x + 4}$
   
   $y = \frac{1}{3}e^{-x} \ln \left| \frac{e^x - 4}{e^x - 1} \right| + Ce^{-x}$; Detailed Solution: Here
7. Look at the First-Order Separable Equations practice problems 3 – 9 and determine which ODE’s, if any, are first-order linear equations. If there are any, solve them using integrating factors.

| Problem #5 is a linear equation since it can be written as \( y' - x^2y = 0 \). The solution is (of course) still \( y = Ce^{x^3/3} \). |

For problems 8-9, solve the initial value problem. Express your answer as an explicit function of \( x \).

8. \( \frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x + x^3}, \) for \( x > 0; \ y(1) = 0 \)

\[
y = \frac{\arctan x}{x} - \frac{\pi}{4x}
\]

9. \( (\cos x) \frac{dy}{dx} + y \sin x = \sin x \cos x, \) for \( -\frac{\pi}{2} < x < \frac{\pi}{2}; \ y(0) = 5 \)

\[
y = (\cos x) \ln (\sec x) + 5 \cos x; \ Detailed \ Solution: \ Here
\]

10. A tank initially contains 7 pounds of salt dissolved in 100 gallons of water. Then, salt water containing 3 pounds of salt per gallon enters the tank at a rate of 8 gallons per minute, and the mixed solution is drained from the tank at a rate of 8 gallons per minute. Let \( y = y(t) \) be the amount of salt in the tank at time \( t \).

(a) Using this information, set up an initial value problem (IVP) whose solution is \( y(t) \).

\[
\begin{aligned}
\frac{dy}{dt} &= 24 - \frac{2y}{25} \\
y(0) &= 7
\end{aligned}
\]

(b) Using integrating factors, solve the IVP from part (a).

\[
y(t) = 300 - 293e^{-2t/25}
\]

(c) Using separation of variables, solve the IVP from part (a).

\[
y(t) = 300 - 293e^{-2t/25}
\]

11. Suppose the saltwater solution in problem #10 is drained from the tank at a rate of 6 gallons per minute.

(a) Set up an initial value problem (IVP) whose solution is \( y(t) \). [Hint: The volume of saltwater is no longer a constant, but rather a function of \( t \).]
\[
\begin{aligned}
\frac{dy}{dt} &= 24 - \frac{3y}{50 + t} \\
y(0) &= 7 \\
\end{aligned}
\]

(b) Using integrating factors, solve the IVP from part (a).
[Note that unlike problem #10 the ODE is no longer separable.]
\[
y(t) = 6(50 + t) - 293(50)^3(50 + t)^{-3}
\]

(c) Suppose that the tank has a capacity of 200 gallons. How much salt is in the tank when it reaches the point of overflowing?

The tank overflows at time \( t = 50 \) minutes.
At that time the amount of salt is \( y(50) = \frac{4507}{8} \approx 563.4 \) pounds

12. Suppose that an object with mass \( m \) falls to the earth with a velocity \( v = v(t) \) and is subjected to the force of gravity as well as air resistance (which is proportional to its velocity). Using Newton’s Second Law it can be shown that
\[
m \frac{dv}{dt} = -mg - kv
\]
where \( g \) is the acceleration due to gravity and \( k \) is some positive constant of proportionality.

(a) Assuming that the object’s initial velocity is \( v_0 \), set up an initial value problem (IVP) whose solution is \( v(t) \).
\[
\begin{aligned}
\frac{dv}{dt} + \frac{k}{m}v &= -g \\
v(0) &= v_0 \\
\end{aligned}
\]

(b) Solve the IVP from part (a).
\[
v(t) = -\frac{gm}{k} + \left( v_0 + \frac{gm}{k} \right) e^{-kt/m}
\]

(c) Evaluate \( \lim_{t \to \infty} v(t) \).
\[
\lim_{t \to \infty} v(t) = -\frac{gm}{k}. \text{ This is known as the terminal velocity of the object and occurs when the opposing forces of air resistance and gravity are equal, causing the object to experience no acceleration.}
\]

13. Consider the simple electrical circuit shown below. An electromotive force (e.g. a generator) produces a voltage of \( V(t) \) volts (V) and a current of \( I(t) \) amperes (A) at time \( t \). The circuit also contains a resistor with a constant resistance of \( R \) ohms (\( \Omega \))
and an inductor with a constant inductance of $L$ henries (H). Such a circuit is called an $RL$ circuit.

Using Ohm’s Law and Kirchoff’s Law it can be shown that

$$L \frac{dI}{dt} + RI = V(t)$$

Suppose that the $RL$ circuit above has a resistance of 6 Ω and an inductance of 3 H. If a generator produces a variable voltage of $V(t) = 9 \sin t$ and the initial current is $I(0) = 2$ A, find $I(t)$.

Hint: Recall to solve an integral of the form $\int e^x \sin x \, dx$, use integration by parts twice.

$$I(t) = -\frac{3}{5} \cos t + \frac{6}{5} \sin t + \frac{13}{5} e^{-2t}$$