The Comparison, Limit Comparison, Ratio, & Root Tests

SUGGESTED REFERENCE MATERIAL:
As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

• Use the following tests to make a conclusion about the convergence of series with no negative terms:
  - Comparison Test
  - Limit Comparison Test
  - Ratio Test
  - Root Test

PRACTICE PROBLEMS:

For problems 1 & 2, apply the Comparison Test to determine if the series converges. Clearly state to which other series you are comparing.

1. \[
\sum_{k=1}^{\infty} \frac{1}{3^k + 5}
\]

\[
\frac{1}{3^k + 5} < \frac{1}{3^k} \text{ for } k \geq 1.
\]

Since \[
\sum_{k=1}^{\infty} \frac{1}{3^k}
\] converges (geometric series, \(r = \frac{1}{3}\)), \[
\sum_{k=1}^{\infty} \frac{1}{3^k + 5}
\] must converge.

2. \[
\sum_{k=3}^{\infty} \frac{1}{3(k - 2)}
\]

\[
\frac{1}{3(k - 2)} = \frac{1}{3k - 6} > \frac{1}{3k} \text{ for } k \geq 3.
\]

Since \[
\sum_{k=3}^{\infty} \frac{1}{3k}
\] diverges (p-series with \(p = 1\)), \[
\sum_{k=3}^{\infty} \frac{1}{3(k - 2)}
\] must diverge as well.

For problems 3 & 4, apply the Limit Comparison Test to determine if the series converges. Clearly state to which other series you are comparing.
3. \[\sum_{k=1}^{\infty} \frac{1}{3(k+2)}\]

\[\lim_{k \to \infty} \frac{\frac{1}{3(k+2)}}{\frac{1}{k}} = \lim_{k \to \infty} \frac{k}{3k+6} = \frac{1}{3},\] which is finite and nonzero.

So since \[\sum_{k=1}^{\infty} \frac{1}{k}\] diverges (Harmonic Series), \[\sum_{k=1}^{\infty} \frac{1}{3(k+2)}\] must diverge as well.

4. \[\sum_{k=2}^{\infty} \frac{1}{3k-5}\]

\[\lim_{k \to \infty} \frac{\frac{1}{3k-5}}{\frac{1}{3k}} = \lim_{k \to \infty} \frac{3k}{3k-5} = 1,\] which is finite and nonzero.

So since \[\sum_{k=2}^{\infty} \frac{1}{3k}\] converges (geometric series, \(r = \frac{1}{3}\)), \[\sum_{k=2}^{\infty} \frac{1}{3k-5}\] must converge.

For problems 5 – 7, apply the Ratio Test to determine if the series converges. If the Ratio Test is inconclusive, apply a different test.

5. \[\sum_{k=0}^{\infty} \frac{1}{k!}\] The series converges by the Ratio Test.

6. \[\sum_{k=0}^{\infty} \frac{1}{3^k}\] The series converges by the Ratio Test. [Of course this just confirms what we already knew as this is a geometric series with \(r = \frac{1}{3}\)].

7. \[\sum_{k=1}^{\infty} \frac{1}{3(k+2)}\] The Ratio Test is inconclusive; however, as shown in #3 above, the series diverges by the Limit Comparison Test.

For problems 8 – 10, apply the Root Test to determine if the series converges. If the Root Test is inconclusive, apply a different test.

8. \[\sum_{k=0}^{\infty} \frac{1}{3^k}\] The series converges by the Root Test. [Of course this just confirms what we already knew as this is a geometric series with \(r = \frac{1}{3}\)].
9. \[ \sum_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right)^k \]

The Root Test is inconclusive; however, as shown in the previous assigned problems, Convergence Tests #7, the series diverges by the Divergence Test.

10. \[ \sum_{k=1}^{\infty} \frac{k}{7^k} \]

The series converges by the Root Test.; Detailed Solution: Here

For problems 11 – 22, apply the Comparison Test, Limit Comparison Test, Ratio Test, or Root Test to determine if the series converges. State which test you are using, and if you use a comparison test, state to which other series you are comparing to.

11. \[ \sum_{k=1001}^{\infty} \frac{1}{\sqrt[3]{k} - 10} \]

The series diverges by the Comparison Test. Compared to \[ \sum_{k=1001}^{\infty} \frac{1}{\sqrt[3]{k}}. \]

12. \[ \sum_{k=1}^{\infty} \frac{4k^2 + 5k}{\sqrt{10 + k^5}} \]

The series diverges by the Limit Comparison Test. Compared to \[ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}. \]

Detailed Solution: Here

13. \[ \sum_{k=0}^{\infty} \frac{2k + 1}{(2k)!} \]

The series converges by the Ratio Test.

14. \[ \sum_{k=1}^{\infty} \frac{k^2 \cos^2 k}{2 + k^5} \]

The series converges by the Comparison Test. Compared to \[ \sum_{k=1}^{\infty} \frac{1}{k^3}. \]

15. \[ \sum_{k=1}^{\infty} \frac{1}{k(5k)} \]

The series converges by the Root Test.
16. \( \sum_{k=0}^{\infty} \frac{2 + 2^k}{3 + 3^k} \)

The series converges by the Limit Comparison Test. Compared to \( \sum_{k=0}^{\infty} \left( \frac{2}{3} \right)^k \).

17. \( \sum_{k=0}^{\infty} \frac{6^{(k+1)}}{k!} \)

The series converges by the Ratio Test.

18. \( \sum_{k=0}^{\infty} \frac{6^k + k}{k! + 6} \) [Hint: Use the result from the previous problem.]

The series converges by the Comparison Test. Compared to \( \sum_{k=0}^{\infty} \frac{6^{(k+1)}}{k!} \). Detailed Solution: Here

19. \( \sum_{k=2}^{\infty} \frac{k^3 - 2}{(k^2 + 1)^2} \)

The series diverges by the Limit Comparison Test. Compared to \( \sum_{k=2}^{\infty} \frac{1}{k} \).

20. \( \sum_{k=1}^{\infty} \frac{\arctan k}{k^{1.5}} \)

The series converges by the Comparison Test. Compared to \( \sum_{k=1}^{\infty} \frac{\pi/2}{k^{1.5}} \).

21. \( \sum_{k=1}^{\infty} \frac{7k}{k^2 + |\sin k|} \)

The series diverges by the Limit Comparison Test. Compared to \( \sum_{k=1}^{\infty} \frac{1}{k} \).

22. \( \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} \)

The series diverges by the Ratio Test.