Cross Product

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Know how to compute the cross product of two vectors in \( \mathbb{R}^3 \).
- Be able to use a cross product to find a vector perpendicular to two given vectors.
- Know how to use a cross product to find areas of parallelograms and triangles.
- Be able to use a cross product together with a dot product to compute volumes of parallelepipeds.

PRACTICE PROBLEMS:

1. For each of the following, compute \( \vec{u} \times \vec{v} \) and verify that it is orthogonal to both \( \vec{u} \) and \( \vec{v} \).
   - (a) \( \vec{u} = \langle 3, -4, 1 \rangle; \vec{v} = \langle 2, -2, 3 \rangle \)
   - (b) \( \vec{u} = \langle 2, -2, 6 \rangle; \vec{v} = \langle -1, 2, -1 \rangle \)
   - (c) \( \mathbf{u} = 2\mathbf{i} + 3\mathbf{k}; \mathbf{v} = \mathbf{i} - \mathbf{j} \)

2. (a) Using appropriate properties of the cross product (Not Determinants), compute \( (\mathbf{i} - \mathbf{j}) \times (\mathbf{j} - \mathbf{i}) \).
   (b) Verify that your answer to part (a) is correct by using determinants.

3. Compute two unit vectors which are normal to the plane which is determined by the points \( A(1, 2, 3), B(6, 4, 7), \) and \( C(1, 5, 2) \).

4. Compute the area of the triangle with vertices \( A(1, 2, 3), B(6, 4, 7), \) and \( C(1, 5, 2) \).

5. Compute \( \|\mathbf{u} \times \mathbf{v}\| \) if \( \|\mathbf{u}\| = 2 \), \( \|\mathbf{v}\| = 5 \), and the angle between \( \mathbf{u} \) and \( \mathbf{v} \) is 30°.

6. The following questions deal with finding the distance from a point to a line:
   - (a) Given three points \( A, B, \) and \( P \) in 3-space as shown in the picture below, explain how you could use the cross product to calculate the distance, \( d \), between the point \( P \) and the line which contains \( A \) and \( B \).
(a) Use your method from part (a) to compute the distance from the point $P(5, 3, 0)$ to the line containing $A(1, 0, 1)$ and $B(2, 3, 1)$. Verify your answer with HW 11.3 #10(b).

7. Consider the parallelepiped with adjacent edges $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle 3, 4, 0 \rangle$, and $\vec{w} = \langle -1, 3, -2 \rangle$.

(a) Compute the volume of the parallelepiped.
(b) Determine the area of the face determined by $\vec{v}$ and $\vec{w}$.
(c) Compute the angle between $\vec{u}$ and the plane containing the face determined by $\vec{v}$ and $\vec{w}$.

8. Multiple Choice: Suppose $u$ and $v$ are non-zero vectors in $\mathbb{R}^3$ and that $u \cdot v = \|u \times v\|$, which of the following is the angle between $u$ and $v$?

(a) 0
(b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$
(d) $\frac{\pi}{3}$
(e) $\frac{\pi}{2}$

9. True or False: Mark each of the following as either true or false. If the statement is false, explain why or provide a counterexample.

(a) The cross product of two vectors in $\mathbb{R}^3$ is anti-commutative; i.e., $v \times w = -(w \times v)$.
(b) $i \times k = j$.
(c) For any vectors $u$ and $v$ in $\mathbb{R}^3$, $\|u \times v\| = \|v \times u\|$.
(d) If $u \times v = 0$, then either $u = 0$ or $v = 0$.
(e) If $u \cdot v = 0$ and $u \times v = 0$, then either $u = 0$ or $v = 0$. 