Parametric Equations of Lines

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

• Be able to find the parametric equations of a line that satisfies certain conditions by finding a point on the line and a vector parallel to the line.

• Know how to determine whether two lines in space are parallel, skew, or intersecting. And, if the lines intersect, be able to determine the point of intersection.

• Know how to determine where a line intersects a surface.

PRACTICE PROBLEMS:

For problems 1-4, compute parametric equations of the line which satisfies the given conditions.

1. The line which passes through the point \((1, 0, -1)\) and is parallel to \(\vec{v} = (1, -2, 0)\).
   \[x = 1 + t, \quad y = -2t, \quad z = -1\]

2. The line which passes through points \(A(3, -6, 6)\) and \(B(2, 0, 7)\).
   \[x = 3 - t, \quad y = -6 + 6t, \quad z = 6 + t\]

3. The line which passes through the point \((-1, 2, 4)\) and is parallel to \(L_1 = \begin{cases} 
  x = 3 - 4t \\
  y = 3 + 2t \\
  z = t
\end{cases}\)
   \[x = -1 - 4t, \quad y = 2 + 2t, \quad z = 4 + t\]

4. The line which passes through the point \((-2, 1, 4)\) and is parallel to both the \(xy\)-plane and the \(xz\)-plane.
   \[x = -2 + t, \quad y = 1, \quad z = 4\]; Detailed Solution: [Here]

5. Is the line which passes through points \(A_1(1, 2, 3)\) and \(B_1(5, 8, 9)\) parallel to the line which passes through points \(A_2(-2, 5, 3)\) and \(B_2(4, 14, 12)\)?
   \[\text{Yes.}\]
6. Find the coordinates of the point at which the line \( L_1 = \begin{cases} x = 3 - 6t \\ y = 3 + 3t \\ z = t \end{cases} \) intersects the given plane:

(a) The \( xy \)-plane.
\[ (x, y, z) = (3, 3, 0) \]
(b) The \( xz \)-plane.
\[ (x, y, z) = (9, 0, -1) \]
(c) The \( yz \)-plane.
\[ (x, y, z) = \left( 0, \frac{9}{2}, \frac{1}{2} \right) \]

7. Find the coordinates of the points in 3-space where the line \( L_1 = \begin{cases} x = t \\ y = 1 + t \\ z = 1 - t \end{cases} \) intersects the sphere \( x^2 + y^2 + z^2 = 29 \).
\[ (x, y, z) = (3, 4, -2) \text{ and } (x, y, z) = (-3, -2, 4); \text{ Detailed Solution: Here} \]

For problems 8-11, determine whether the given lines intersect, are parallel, or are skew. If the lines intersect, find the point of intersection.

8. \( L_1 : x = 2 + 3t, y = 1 - 2t, z = 4 + 5t \)
\( L_2 : x = 3 - 6t, y = -2 + 4t, z = -1 - 10t \)
\[ \text{The lines are parallel.} \]

9. \( L_1 : x = 1, y = t, z = 2 - t \)
\( L_2 : x = 2 + 3t, y = 4 - 3t, z = t \)
\[ \text{The lines are skew.} \]

10. \( L_1 : x = 1 - 2t, y = 14 + t, z = 5 - t \)
\( L_2 : x = t, y = 4 + 3t, z = 3 + t \)
\[ \text{The lines intersect at the point } (x, y, z) = (3, 13, 6); \text{ Detailed Solution: Here} \]

11. \( L_1 : x = 2 + 5t, y = 4 - t, z = t + 1 \)
\( L_2 : x = 3 + 6t, y = 1 - t, z = t \)
\[ \text{The lines are skew.} \]
12. Verify that the following lines are parallel. Then compute the distance between them.  
(Hint: See HW 11.3 #10 or 11.4 #6.) 

\[ L_1 : x = 5 + 3t, y = 3 + 9t, z = 0 \]
\[ L_2 : x = 1 + t, y = 3t, z = 1 \]

The lines are parallel because \( \langle 3, 9, 0 \rangle = 3 \langle 1, 3, 0 \rangle \). The distance between the lines is 
\[ d = \sqrt{\frac{91}{10}} \]

13. Two bugs are walking along lines in 3-space. At time \( t \), bug 1’s position is the point \((x, y, z)\) on the line \( L_1 = \begin{cases} x = 1 + 2t \\ y = 3 + 5t \\ z = 5 + 2t \end{cases} \) and bug 2’s position is the point \((x, y, z)\) on the line \( L_2 = \begin{cases} x = t \\ y = 11 - t \\ z = 4 + t \end{cases} \)

(a) Compute the distance between the bugs’ initial positions. 

Bug 1’s initial position is \((x, y, z) = (1, 3, 5)\) and Bug 2’s initial position is \((x, y, z) = (0, 11, 4)\). The distance between these two points is \( \sqrt{66} \)

(b) At which point in space will the bugs’ paths intersect? (Note: the paths may not intersect at the same moment in time.)

The paths intersect at the point \((x, y, z) = (3, 8, 7)\)

14. Consider the point \( P(5, 3, 0) \) and the line \( L \) which contains points \( A(1, 0, 1) \) and \( B(2, 3, 1) \). This problem will show you another way to find the distance \( d \) between the point \( P \) and the line \( L \).

(a) Compute an equation of line \( L \).

\[ \vec{\ell}(t) = \langle 1 + t, 3t, 1 \rangle \]

(b) Compute a function \( f(t) \) which gives the distance from the point \( P \) to an arbitrary point on the line.
Your answer to this part depends on your parametric equations from part (a).
Using the parameterization given the distance from $P$ to an arbitrary point on
line $L$ is given by $f(t) = \sqrt{(4 - t)^2 + (3 - 3t)^2 + 1}$.

(c) The distance from the point $P$ to line $L$ is the shortest distance. Calculate the
value of $t$ which minimizes the distance from the point $P$ to line $L$; that is,
calculate the value of $t$ which minimizes $f(t)$ from part (b).

\[
\begin{align*}
t &= \frac{13}{10}; \text{ again, this depends on your parameterization of the line.}
\end{align*}
\]

(d) Compute the distance from the point $P(5, 3, 0)$ to line $L$ by calculating the dis-
tance from this $P$ to the point on your the line which corresponds to your value
of $t$ from part (c). Verify your answer with HW 11.3 #10(b).

\[
\begin{align*}
d &= \sqrt{\frac{91}{10}}
\end{align*}
\]