Planes

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

• Be able to find the equation of a plane that satisfies certain conditions by finding a point on the plane and a vector normal to the plane.

• Know how to find the parametric equations of the line of intersection of two (non-parallel) planes.

• Be able to find the (acute) angle of intersection between two planes.

PRACTICE PROBLEMS:

1. For each of the following, find an equation of the plane indicated in the figure.

   (a) \(6x + 4y + 3z = 12\); (b) \(3x + 2y = 6\)

For problems 2-6, determine whether the following are parallel, perpendicular, or neither.

2. Plane \(P_1 : 5x - 3y + 4z = -1\) and plane \(P_2 : 2x - 2y - 4z = 9\)

   The planes are perpendicular.
3. Plane \( P_1 : 3x - 2y + z = -3 \) and plane \( P_2 : 5x + y - 6z = 8 \)

The planes are neither parallel nor perpendicular; Detailed Solution: Here

4. Plane \( P_1 : 3x - 2y + z = -3 \) and plane \( P_2 : -6x + 4y - 2z = 1 \)

The planes are parallel.

5. Plane \( P : 5x - 3y + 4z = -1 \) and line \( \vec{\ell}(t) = (2 + 2t, 3 - 2t, 5 - 4t) \)

The plane and the line are parallel.

6. Plane \( P : 5x - 3y + 4z = -1 \) and line \( \vec{\ell}(t) = \left< 2 + \frac{5}{2}t, 3 - \frac{3}{2}t, 5 + 2t \right> \)

The plane and the line are perpendicular.

7. Give an example of a plane, \( P \), and a line, \( L \), which are neither parallel nor perpendicular to each other.

Suppose your line has the form \( \vec{\ell}(t) = \vec{\ell}_0 + t \vec{v} \) and that your plane has \( \vec{n} \) as a normal vector. Then all possible answers are those for which \( \vec{v} \parallel \vec{n} \) (i.e., \( \vec{v} = c\vec{n} \) for any scalar \( c \)) and \( \vec{v} \not\perp \vec{n} \) (i.e., \( \vec{v} \cdot \vec{n} \neq 0 \)). The first condition ensures that \( L \) and \( P \) are not perpendicular; the second condition ensures that \( L \) and \( P \) are not parallel.

For problems 8-13, find an equation of the plane which satisfies the given conditions.

8. The plane which passes through the point \( P(1, 2, 3) \) and which has a normal vector of \( \vec{n} = 4\vec{i} - 2\vec{j} + 6\vec{k} \).

\[ 4(x - 1) - 2(y - 2) + 6(z - 3) = 0 \]

9. The plane which passes through \( P(-2, 0, 1) \) and is perpendicular to the line \( \vec{\ell}(t) = (1, 2, 3) + t(3, -2, 2) \).

\[ 3(x + 2) - 2y + 2(z - 1) = 0 \]

10. The plane which passes through points \( A(1, 2, 3), B(2, -1, 5) \) and \( C(-1, 3, 3) \).

\[ -2(x - 1) - 4(y - 2) - 5(z - 3) = 0 \]

11. The plane which passes through \( A(1, 2, 3) \) and is parallel to the plane \( 3x - 5y + z = 2 \).

\[ 3(x - 1) - 5(y - 2) + 1(z - 3) = 0 \]

12. The plane which passes through \( A(-2, 1, 5) \) and is perpendicular to the line of intersection of \( P_1 : 3x + 2y - z = 5 \) and \( P_2 : -y + z = 7 \).

\[ 1(x + 2) - 3(y - 1) - 3(z - 5) = 0 \]; Detailed Solution: Here
13. The plane which contains the point \( A(-2, -1, 3) \) and which contains the line \( L : x = 1 + t, y = 3 - 2t, z = 4t \).
\[
2(x + 2) - 3(y + 1) - 2(z - 3) = 0
\]

14. Consider the planes \( P_1 : x + y + z = 7 \) and \( P_2 : 2x + 4z = 6 \).

(a) Compute an equation of the line of intersection of \( P_1 \) and \( P_2 \).
One parametric equation of the line of intersection is \( L : x = 3 - 2t, y = 4 + t, z = t \)

(b) Compute an equation of the plane which passes through the point \( A(1, 2, 3) \) and contains the line of intersection of \( P_1 \) and \( P_2 \).
\[
5(x - 1) + 4(y - 2) + 6(z - 3) = 0
\]

15. Find the acute angle of intersection of \( P_1 : 3x - 2y + 5z = 0 \) and \( P_2 : -x - y + 2z = 3 \).
\[
\cos^{-1}\left( \frac{9}{\sqrt{38}\sqrt{6}} \right)
\]

16. Find the acute angle of intersection of \( P_1 : 3x - 2y - 5z = 0 \) and \( P_2 : -x - y + 2z = 3 \).
\[
\pi - \cos^{-1}\left( \frac{-11}{\sqrt{38}\sqrt{6}} \right); \text{ Detailed Solution: Here}
\]

17. Consider the plane which passes through the point \( Q \) and whose normal vectors are parallel to \( n \). And, let \( P \) be another point in space, as illustrated below.

(a) Show that the distance between the point \( P \) and the given plane is \( d = \frac{|QP \cdot n|}{\|n\|} \).
\[
d = \|\text{Proj}_n QP\| = \| \left( \frac{QP \cdot n}{\|n\|^2} \right) n \| = \frac{|QP \cdot n|}{\|n\|^2} \|n\| = \frac{|QP \cdot n|}{\|n\|}
\]
(b) Use this method to compute the distance between the point \( P(2, -1, 4) \) and the plane \( x + 2y + 3z = 5 \).

\[
d = \frac{7}{\sqrt{14}}
\]

18. Consider planes \( P_1 : 2x - 4y + 5z = -2 \) and \( P_2 : x - 2y + \frac{5}{2}z = 5 \).

(a) Verify that \( P_1 \) and \( P_2 \) are parallel.

\[\mathbf{n}_1 = \langle 2, -4, 5 \rangle \text{ is normal to plane } P_1.\]

\[\mathbf{n}_2 = \langle 1, -2, \frac{5}{2} \rangle \text{ is normal to plane } P_2.\]

Since \( \mathbf{n}_1 = 2\mathbf{n}_2 \), we have that \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) are parallel. And, because these normal vectors are parallel, the planes \( P_1 \) and \( P_2 \) are parallel, too.

(b) Compute the distance between \( P_1 \) and \( P_2 \). (Hint: See the previous problem.)

\[
d = \frac{12}{\sqrt{45}}; \text{ Detailed Solution: Here}\]