Multivariable Chain Rule

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

• Be able to compute partial derivatives with the various versions of the multivariate chain rule.
• Be able to compare your answer with the direct method of computing the partial derivatives.

PRACTICE PROBLEMS:

1. Find \( \frac{dz}{dt} \) by using the Chain Rule. Check your answer by expressing \( z \) as a function of \( t \) and then differentiating.

   (a) \( z = 2x - y, \ x = \sin t, \ y = 3t \)

   \[ \frac{dz}{dt} = 2 \cos t - 3 \]

   (b) \( z = x \sin y, \ x = e^t, \ y = \pi t \)

   \[ \frac{dz}{dt} = e^t \sin (\pi t) + \pi e^t \cos (\pi t) \]

   (c) \( z = xy + y^2, \ x = t^2, \ y = t + 1 \)

   \[ \frac{dz}{dt} = 3t^2 + 4t + 2 \]

   (d) \( z = \ln \left( \frac{x^2}{y} \right), \ x = e^t, \ y = t^2 \)

   \[ \frac{dz}{dt} = 2 - \frac{2}{t} \]

2. Suppose \( w = x^2 + y^2 + 2z^2, \ x = t + 1, \ y = \cos t, \ z = \sin t \). Find \( \frac{dw}{dt} \) using the Chain Rule. Check your answer by expressing \( w \) as a function of \( t \) and then differentiating.

   \[ \frac{dw}{dt} = 2t + 2 + 2 \sin t \cos t; \text{ Detailed Solution: Here} \]
3. Suppose \( f \) is a differentiable function of \( x \) & \( y \), and define \( g(u, v) = f(3u - v, u^2 + v) \).

Use the table of values shown below to calculate \( \frac{\partial g}{\partial u} \bigg|_{(u,v)=(2,-1)} \) and \( \frac{\partial g}{\partial v} \bigg|_{(u,v)=(2,-1)} \).

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f)</th>
<th>(g)</th>
<th>(f_x)</th>
<th>(f_y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2, -1))</td>
<td>6</td>
<td>-7</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>((7, 3))</td>
<td>4</td>
<td>2</td>
<td>-3</td>
<td>5</td>
</tr>
</tbody>
</table>

Hint: Decompose \( f(3u - v, u^2 + v) \) into \( f(x, y) \) where \( x = 3u - v \) and \( y = u^2 + v \).

\( g_u(2, -1) = 11; g_v(2, -1) = 8 \)

4. Find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) by using the appropriate Chain Rule.

(a) \( w = x y \sin (z^2), \ x = s - t, \ y = s^2, \ z = t^2 \)

\[
\frac{\partial w}{\partial s} = s^2 \sin (t^4) + 2s(s-t)\sin (t^4); \quad \frac{\partial w}{\partial t} = -s^2 \sin (t^4) + 4s^2t^3(s-t)\cos (t^4)
\]

(b) \( w = x y + y z, \ x = s + t, \ y = st, \ z = s - 2t \)

\[
\frac{\partial w}{\partial s} = 4st - t^2; \quad \frac{\partial w}{\partial t} = 2s^2 - 2st
\]

5. Suppose that \( J = f(x, y, z, w) \), where \( x = x(r, s, t), \ y = y(r, t), \ z = z(r, s) \) and \( w = w(s, t) \). Use the Chain Rule to find \( \frac{\partial J}{\partial r}, \frac{\partial J}{\partial s}, \) and \( \frac{\partial J}{\partial t} \).

\[
\frac{\partial J}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial r};
\]
\[
\frac{\partial J}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial s};
\]
\[
\frac{\partial J}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial t}
\]
6. Suppose \( g = f(u - v, v - w, w - u) \). Show that \( \frac{\partial g}{\partial u} + \frac{\partial g}{\partial v} + \frac{\partial g}{\partial w} = 0 \).

We can express \( g \) as \( g = f(x, y, z) \), where \( x = u - v, y = v - w, \) and \( z = w - u \). Then, we compute the partial derivatives of \( g \) with respect to \( u, v, \) and \( w \).

\[
\begin{align*}
\frac{\partial g}{\partial u} &= f_x(u - v, v - w, w - u) - f_z(u - v, v - w, w - u) \\
\frac{\partial g}{\partial v} &= -f_x(u - v, v - w, w - u) + f_y(u - v, v - w, w - u) \\
\frac{\partial g}{\partial w} &= -f_y(u - v, v - w, w - u) + f_z(u - v, v - w, w - u)
\end{align*}
\]

Summing these three partial derivatives yields 0.

7. Suppose \( u = u(x, y), v = v(x, y), x = r \cos \theta, \) and \( y = r \sin \theta \).

(a) Calculate \( \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r} \), and \( \frac{\partial v}{\partial \theta} \)

\[
\begin{align*}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta; \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial u}{\partial x} \sin \theta + r \frac{\partial u}{\partial y} \cos \theta; \\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta; \quad \frac{\partial v}{\partial \theta} = -r \frac{\partial v}{\partial x} \sin \theta + r \frac{\partial v}{\partial y} \cos \theta
\end{align*}
\]

(b) Suppose that \( u(x, y) \) and \( v(x, y) \) satisfy the Cauchy-Riemann Equations:

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{align*}
\]

Use this along with part (a) to derive the polar form of the Cauchy-Riemann Equations:

\[
\begin{align*}
\frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \\
\frac{\partial u}{\partial \theta} &= -r \frac{\partial v}{\partial r}
\end{align*}
\]
From part (a), we know that $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$. We apply the Cauchy Riemann Equations to $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Thus,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta = \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta = \frac{1}{r} \left( r \frac{\partial v}{\partial y} \cos \theta - r \frac{\partial v}{\partial x} \sin \theta \right) = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

A similar argument yields the second Cauchy Riemann Equation in polar coordinates.