The Gradient & Directional Derivatives

SUGGESTED REFERENCE MATERIAL:
As you work through the problems listed below, you should reference Chapter 13.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute a gradient vector, and use it to compute a directional derivative of a given function in a given direction.

- Be able to use the fact that the gradient of a function $f(x, y)$ is perpendicular (normal) to the level curves $f(x, y) = k$ and that it points in the direction in which $f(x, y)$ is increasing most rapidly.

PRACTICE PROBLEMS:

For problems 1-3, compute the directional derivative of $f$ at the point $P$ in the direction of $\vec{v}$.

1. $f(x, y) = x^4 - y^4$; $P(0, -2)$; $\vec{v} = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j$
   
   $\frac{32}{\sqrt{2}}$

2. $f(x, y) = y \sin x$; $P\left(\frac{\pi}{2}, 1\right)$; $\vec{v} = \langle 1, -1 \rangle$
   
   $\frac{-1}{\sqrt{2}}$

3. $f(x, y, z) = e^x \cos (yz)$ at $P = (1, \pi, 0)$, $\vec{v} = -2i + j - 3k$
   
   $\frac{-2e}{\sqrt{14}}$

4. Find the directional derivative of $g(x, y, z) = z \ln (x + y)$ at $P(0, 1, -2)$ in the direction from $P$ to $Q(1, 3, 2)$.
   
   $\frac{-6}{\sqrt{21}}$; Detailed Solution: [Here]
5. Find the directional derivative of \( f(x, y) = \frac{y^2}{x+y} \) at the point \((-1, -1)\) in the direction of a vector which makes a counterclockwise angle \( \theta = \frac{\pi}{4} \) with the positive \( x \)-axis.

\[ \frac{\sqrt{2}}{4} \]

6. Suppose \( f(x, y) = \tan (xy) \). Find a unit vector \( \mathbf{u} \) such that \( D_u f(1, \pi) = 0 \).

\[ \mathbf{u} = \left\langle \frac{1}{\sqrt{\pi^2 + 1}}, -\frac{\pi}{\sqrt{\pi^2 + 1}} \right\rangle \text{ or } \mathbf{u} = \left\langle -\frac{1}{\sqrt{\pi^2 + 1}}, \frac{\pi}{\sqrt{\pi^2 + 1}} \right\rangle \]

7. Suppose that \( f(x, y, z) \) is a differentiable function. Let \( f_x(1, 1, 2) = 5, f_y(1, 1, 2) = -1, \) and \( f_z(1, 1, 2) = 0 \). What is the directional derivative of \( f(x, y, z) \) at \((1, 1, 2)\) in the direction of \( \mathbf{a} = \langle -3, 0, 4 \rangle \)?

\( -3 \)

8. Suppose \( D_u f(3, -2) = 1 \) and \( D_v f(3, -2) = 2 \) where \( \mathbf{u} = \frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j} \) and \( \mathbf{v} = -\frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j} \). Compute \( f_x(3, -2) \) and \( f_y(3, -2) \).

\( f_x(3, -2) = -\frac{5}{8}; f_y(3, -2) = \frac{5}{2} \)

For problems 9-11, find the gradient of \( f \) at the given point.

9. \( f(x, y) = 3xy - y^2x^3 \) at \((1, -1)\)

\[ \nabla f(1, -1) = -6 \mathbf{i} + 5 \mathbf{j} \]

10. \( f(x, y) = \cos (2x - y^2) \) at \((\pi/4, 0)\)

\[ \nabla f \left( \frac{\pi}{4}, 0 \right) = \langle -2, 0 \rangle \]

11. \( f(x, y, z) = 4xyz - y^2z^3 + 4z^3y \) at \((2, 3, 1)\)

\[ \nabla f(2, 3, 1) = 12 \mathbf{i} + 6 \mathbf{j} + 33 \mathbf{k} \]

12. For each of the following, determine the maximum value of the directional derivative at the given point as well as a unit vector in the direction in which the maximum value occurs.

(a) \( g(x, y) = e^{xy}; \) \( P(1, 3) \)

The maximum value of the directional derivative of \( g \) at \( P \) is \( e^9 \sqrt{117} \) which occurs in the direction of \( \mathbf{u} = \left\langle \frac{9}{\sqrt{117}}, \frac{6}{\sqrt{117}} \right\rangle \).
(b) \( w = \sqrt{4 - x^2 - y^2 - z^2}; \ P(1, -1, 0) \)

The maximum value of the directional derivative of \( w \) at \( P \) is 1 which occurs in the direction of \( \mathbf{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle \).

13. The temperature at the point \((x, y, z)\) in a room is \( T(x, y, z) = \frac{xz}{x^2 + y^2} \). Find the direction in which the temperature increases most rapidly at the point \((-3, 4, 1)\).

\[
\frac{7}{625} \mathbf{i} + \frac{24}{625} \mathbf{j} - \frac{3}{25} \mathbf{k}
\]

14. Compute a unit vector in the direction in which \( f(x, y, z) = x^3yz^2 \) decreases most rapidly at \( P(2, -1, 1) \); and, find the rate of change of \( f \) at \( P \) in that direction.

The direction in which \( f \) decreases most rapidly is \( \mathbf{u} = \left\langle \frac{3}{\sqrt{29}}, -\frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle \). And, the rate of change in this direction is \(-4\sqrt{29}\). Detailed Solution: [Here]

For problems 15-16, sketch the level curve of \( f(x, y) \) which passes through the given point \( P \). Then draw the gradient of \( f \) at \( P \) on the same axes.

15. \( f(x, y) = 20 - 5x + y; \ P = (3, 5) \)

Point \( P \) is on the level curve \( f(x, y) = 10 \), i.e., \( y = 5x - 10 \); \( \nabla f(3, 5) = \langle -5, 1 \rangle \).
16. \( f(x, y) = x^2 + y^2; \ P \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \)

Point \( P \) is on the level curve \( f(x, y) = 1 \), i.e., \( x^2 + y^2 = 1 \); \( \nabla f \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) = \left( \sqrt{2}, \sqrt{2} \right) \).

17. The graph shown below depicts some level curves of an unspecified function \( f(x, y) \).

Which of the vectors is most likely to be \( \nabla f \) at \( P \)? Explain your reasoning.

\( \vec{d} \). \( \nabla f(P) \) should point in the direction of greatest increase and it should be normal to point \( P \) on the level curve of \( f(x, y) \).