Tangent Planes & Normal Lines

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.7 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute an equation of the tangent plane at a point on the surface $z = f(x, y)$.
- Given an implicitly defined level surface $F(x, y, z) = k$, be able to compute an equation of the tangent plane at a point on the surface.
- Know how to compute the parametric equations (or vector equation) for the normal line to a surface at a specified point.
- Be able to use gradients to find tangent lines to the intersection curve of two surfaces. And, be able to find (acute) angles between tangent planes and other planes.

PRACTICE PROBLEMS:

For problems 1-4, find two unit vectors which are normal to the given surface $S$ at the specified point $P$.

1. $S : 2x - y + z = -7; \ P(-1, 2, -3)$
2. $S : x^2 - 3y + z^2 = 11; \ P(-1, -2, 2)$
3. $S : z = y^4; \ P(3, -1, 1)$
4. $S : z = 2 - x^2 \cos(xy); \ P\left(-1, \frac{\pi}{2}, 2\right)$

For problems 5-9, compute equations of the tangent plane and the normal line to the given surface at the indicated point.

5. $S : \ln (x + y + z) = 2; \ P(-1, e^2, 1)$
6. $S : x^2 + y^2 + z^2 = 1; \ P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$
7. $S : z = \arcsin\left(\frac{x}{y}\right); \ P\left(-1, -\sqrt{2}, \frac{\pi}{4}\right)$
8. \( S : x^2 - xy + z^2 = 9; \; P(2, 2, 3) \)

9. \( S : z = x \cos (x + y); \; P \left( \frac{\pi}{2}, \frac{\pi}{3}, -\frac{\sqrt{3}\pi}{4} \right) \)

10. Consider the surfaces \( S_1 : x^2 + y^2 = 25 \) and \( S_2 : z = 2 - x \)

   (a) Find an equation of the tangent line to the curve of intersection of \( S_1 \) and \( S_2 \) at the point \((3, 4, -1)\).

   (b) Find the acute angle between the planes which are tangent to the surfaces \( S_1 \) and \( S_2 \) at the point \((3, 4, -1)\).

11. Consider the surfaces \( S_1 : z = x^2 - y^2 \) and \( S_2 : y^2 + z^2 = 10 \)

   (a) Find an equation of the tangent line to the curve of intersection of \( S_1 \) and \( S_2 \) at the point \((2, 1, 3)\).

   (b) Find the acute angle between the planes which are tangent to the surfaces \( S_1 \) and \( S_2 \) at the point \((2, 1, 3)\).

12. Find all points on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 72 \) where the tangent plane is parallel to the plane \( 4x + 4y + 12z = 3 \).

13. Find all points on the hyperboloid of 1 sheet \( x^2 + y^2 - z^2 = 9 \) where the normal line is parallel to the line which contains points \( A(1, 2, 3) \) and \( B(7, 6, 5) \).

14. Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that the sphere \( x^2 + y^2 + z^2 = 1 \) and the cone \( z^2 = x^2 + y^2 \) are orthogonal at all points of intersection. (HINT: Assume that the surfaces intersect at the arbitrary point \((x_0, y_0, z_0)\).)

15. Show that every plane which is tangent to the cone \( z^2 = x^2 + y^2 \) must pass through the origin. (HINT: Compute the equation of the plane which is tangent to the surface at the point \( P_0(x_0, y_0, z_0) \) and see what happens.)