Double Integrals Over General Regions

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 14.2 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute double integral calculations over rectangular regions using partial integration.
- Know how to inspect an integral to decide if the order of integration is easier one way \((y \text{ first, } x \text{ second})\) or the other \((x \text{ first, } y \text{ second})\).
- Know how to use a double integral to calculate the volume under a surface or find the area or a region in the \(xy\)-plane.
- Know how to reverse the order of integration to simplify the evaluation of a double integral.

PRACTICE PROBLEMS:

1. Consider the region \(R\) shown below which is enclosed by \(y = x^3\), \(y = 0\) and \(x = 1\).

![Diagram of the region R](image)

Fill in the missing limits of integration.

(a) \[
\iint_R f(x, y) \, dA = \int_0^1 \int_{x^3}^0 f(x, y) \, dy \, dx
\]

\[
\iint_R f(x, y) \, dA = \int_0^1 \int_{0}^{x^3} f(x, y) \, dy \, dx
\]
2. Consider the region $R$ shown below which is enclosed by $y = \sqrt{4-x^2}$ and $y = \frac{1}{2}(x+2)$.

(a) Set up $\iiint_{R} f(x, y) \, dA$ with the order of integration as $dy \, dx$

$$\int_{-2}^{6/5} \int_{\frac{x}{2}+2}^{\sqrt{4-x^2}} f(x, y) \, dy \, dx$$

(b) Set up $\iiint_{R} f(x, y) \, dA$ with the order of integration as $dx \, dy$

$$\int_{0}^{8/5} \int_{-\sqrt{4-y^2}}^{2y-2} f(x, y) \, dx \, dy + \int_{8/5}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y) \, dx \, dy$$

For problems 3-7, evaluate the iterated integral. For some problems, it may be helpful to switch the order of integration.

3. $\int_{1}^{2} \int_{-x}^{x} (y^2 + 3xy + x^3) \, dy \, dx$

4. $\int_{0}^{\pi/3} \int_{0}^{\sin x} y \cos x \, dy \, dx$
5. \[ \int_{0}^{1} \int_{0}^{x^3} \sqrt{1 - x^4} \, dy \, dx \]

\[ \frac{1}{6} \]

6. \[ \int_{0}^{1} \int_{y}^{1} \sqrt{1 - x^2} \, dx \, dy \]

\[ \frac{1}{3} \]

7. \[ \int_{0}^{\sqrt{\pi}/2} \int_{2y}^{\sqrt{\pi}} \sin(x^2) \, dx \, dy \]

\[ \frac{1}{2} \]

8. Evaluate \( \iint_{R} (4x - 3y) \, dA \) where \( R \) is the region enclosed by the circle \( x^2 + y^2 = 1 \).

\[ 0 \]

9. Evaluate \( \iint_{R} xy^2 \, dA \) where \( R \) is the triangular region enclosed by \( y = 3x \), \( y = \frac{x}{2} \), and \( y = 1 \).

\[ \frac{7}{18} \]

10. Let \( R \) be the region enclosed by \( y = x^2 \) and \( y = 2x + 3 \).

   (a) Set up a double integral (or double integrals) with the order of integration as \( dy \, dx \) which represents the area of \( R \).

   \[ \int_{-1}^{3} \int_{x^2}^{2x+3} 1 \, dy \, dx \]

   (b) Set up a double integral (or double integrals) with the order of integration as \( dx \, dy \) which represents the area of \( R \).

   \[ \int_{0}^{\sqrt{y}} \int_{1}^{\sqrt{y}} \frac{1}{2} \, dx \, dy + \int_{-\sqrt{y}}^{0} \int_{1}^{\sqrt{y}} 1 \, dx \, dy \]

   (c) Compute the area of \( R \).

   \[ \frac{32}{3} \]
11. Use a double integral to find the volume of the solid in the first octant which is enclosed by the surface $3x + 6y + 2z = 12$ and the coordinate planes.

12. Consider the solid that enclosed by the cylinder $\frac{x^2}{9} + y^2 = 1$ and the planes $z = 0$ and $x + 2y + z = 4$. Use a double integral to compute the volume of this wedge.

13. Let $R$ be the region in the first quadrant of the $xy$ plane which is enclosed by $y = \sqrt{x}$, $x = 0$ and $y = 1$. Compute the volume of the solid which is bounded above by $z = xe^{x/y^2}$ and has $R$ as its base.