Preliminaries

QUICK REVIEW: Incoming Calculus I students should be familiar with the following facts about the rectangular coordinate system.

- Let $A$, $B$, and $C$ be the lengths of the three sides of a right triangle where $C$ is the length of the hypotenuse. Then, the Pythagorean Theorem says that $A^2 + B^2 = C^2$.
- The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A circle with a radius of $(h, k)$ and a radius of $r$ has an equation of $(x-h)^2 + (y-k)^2 = r^2$.
- You should be familiar with basic area, volume, and perimeter formulas from geometry.

PRACTICE PROBLEMS:

1. Use the Pythagorean Theorem and the distance formula to show that the points $(4, 0)$, $(2, 1)$, and $(-1, -5)$ are vertices of a right triangle.

2. For each of the following, determine an equation of the circle which has the given characteristics:

   (a) Center: $(2, -1)$; Radius: 4
   (b) Endpoints of a diameter: $(-3, -2)$ and $(-7, 8)$

3. A ”Slow Moving Vehicle” sign has the shape of an equilateral triangle. The sign has a perimeter of 120 centimeters.

   (a) Find the length of each side of the sign.
   (b) Find the area of the sign.

4. A rectangular room is 1.5 times as long as it is wide. Its perimeter is 25 meters. Find the dimensions of the room.


Lines

QUICK REVIEW: Incoming Calculus I students should be familiar with the following facts about lines.

- The **slope** of a line through the points \((x_1, y_1)\) and \((x_2, y_2)\) is given by \(m = \frac{y_2 - y_1}{x_2 - x_1}\).

- The slope of any horizontal line is 0 and the slope of any vertical line is undefined.

- The **Point-Slope Form** of a line with a slope of \(m\) which passes through the point \((x_1, y_1)\) is \(y - y_1 = m(x - x_1)\).

- The **Slope-Intercept Form** of a line with a slope of \(m\) and a \(y\)-intercept of \(b\) is \(y = mx + b\). Note: a \(y\)-intercept of \(b\) means that the graph passes through the point \((0, b)\).

- The equation of the horizontal line which passes through the point \((a, b)\) is \(y = b\) and the equation of the vertical line which passes through the point \((a, b)\) is \(x = a\).

- Two lines are parallel is thy have the same slope OR they are both vertical.

- Two lines are perpendicular is the product of their slopes is \(-1\) (i.e., their slopes are negative-reciprocals) OR one is vertical whereas the other is horizontal.

WORKED EXAMPLES

1. Find an equation of the line which passes through the point \((1, -2)\) and which has a slope of 5.
   
   **Solution #1:** We are given enough information to use slope form. Specifically, we know that \((x_1, y_1) = (1, -2)\) and \(m = 5\). Thus \(y - (-2) = 5(x - 1)\). If we simplify this, we get \(y = 5x - 7\).

   **Solution #2:** We could have used slope-intercept form instead. Specifically, since \(m = \frac{5}{1}\), we have that \(y = 5x + b\). The equation must hold for all points on the line; so, if we substitute in \((x, y) = (1, -2)\), we can solve for \(b\). I.e., \(-2 = 5(1) + b\) which gives us that \(b = -7\). Thus, the equation of the specified line is \(y = 5x - 7\), as before.

2. Find an equation of the line which passes through \((1, 3)\) and is perpendicular to the line \(3x + 6y = 5\).

   **Solution:** First, we find the slope of the given line \(3x + 6y = 5\). One way to do this is to re-write the equation in slope-intercept form: \(y = -\frac{1}{2}x + \frac{5}{6}\). We now see that the given line has a slope of \(-\frac{1}{2}\) and realize that any line perpendicular to it must have a slope of 2. Then, the equation of the requested line in point-slope form is \(y - 3 = 2(x - 1)\), which is equivalent to the slope-intercept form of \(y = 2x - 1\).
PRACTICE PROBLEMS:

1. For each of the following, write the equation of the line in slope-intercept form (where appropriate) which satisfies the given condition.

   (a) The line which passes through (1, 2) and (5, 7).
   (b) The line which has an x intercept of 5 and a y intercept of 3.
   (c) The line which is passes through (−5, 9) and (4, 9).
   (d) The line which is parallel to \( y = 4x - 7 \) and passes through \( \left( \frac{1}{4}, 2 \right) \).
   (e) The line which is perpendicular to \( y = 4x - 7 \) and passes through \( \left( \frac{1}{4}, 2 \right) \).
   (f) The line which is perpendicular to \( y = 3 \) and passes through the (1, 0)

Functions

QUICK REVIEW: Incoming Calculus I students should be familiar with the following facts about functions.

- A function from a set \( A \) to a set \( B \) is a relation (rule) which assigns to each element \( x \) in \( A \) exactly one element \( y \) in the set \( B \). Notation: \( y = f(x) \)
- The domain of a function is the set of all allowable inputs (set \( A \) in the above definition). The range of a function is the set of all resulting output values (set \( B \) in the above definition).
- The Vertical Line Test says that a given graph represents a function is any vertical line intersects the graph at most once. So, if you can draw a vertical line which intersects a given graph two or more times, the graph does not represent a function.
- Let \( f \) and \( g \) be functions. We define:
  - \((f + g)(x) = f(x) + g(x)\)
  - \((f - g)(x) = f(x) - g(x)\)
  - \((fg)(x) = f(x)g(x)\)
  - \(\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}\)
- The composition of the function \( f \) with \( g \) is denoted \( f \circ g \). This function is defined as \((f \circ g)(x) = f(g(x))\). In words, the output of the first function (\( g \) in this case) is the input of a second function (\( f \) in this case). Notice that we can illustrate composition \( f \circ g \) as follows:
\[ x \rightarrow [g] \rightarrow g(x) \rightarrow [f] \rightarrow f(g(x)) \]

CAUTION: In general, \( f \circ g \neq g \circ f \). In words, the order of the composition is usually important.

- Functions \( f \) and \( g \) are inverse functions if \( f(g(x)) = x \) for all \( x \) in the domain of \( g \) AND \( g(f(x)) = x \) for all \( x \) in the domain of \( f \). Notation: the inverse function of \( f \) is written \( f^{-1}(x) \).

- The domain of \( f \) is the range of \( f^{-1} \) and the range of \( f \) is the domain of \( f^{-1} \).

- the **Horizontal Line Test** says that a function \( f \) has an inverse function \( f^{-1} \) if and only if any horizontal line intersects the graph of \( f \) at most once. If you can draw a horizontal line which intersects the graph of \( f \) more than once, then \( f^{-1} \) does not exist on the given domain.

PRACTICE PROBLEMS:

1. Which of the following graphs represents \( y \) as a function of \( x \)?

(a) ![Graph A](image)

(b) ![Graph B](image)

(c) ![Graph C](image)

(d) ![Graph D](image)

For numbers 2-3, evaluate the function at the given input.

2. \( f(x) = 2x - 3 \)
(a) \( f(1) \)
(b) \( f(-3) \)
(c) \( f(x-1) \)

3. \( f(x) = \begin{cases} 
3x - 1 & \text{if } x < -1 \\
4 & \text{if } -1 \leq x \leq 1 \\
x^2 & \text{if } x > 1 
\end{cases} \)

(a) \( f(-2) \)
(b) \( f\left(-\frac{1}{2}\right) \)
(c) \( f(3) \)
(d) \( f(3t) \)

4. Let \( f(x) = x^2 - 2x \). Compute and simplify the difference quotient \( \frac{f(x+h) - f(x)}{h} \).

5. Let \( f(x) = \frac{1}{x^2} \). Compute and simplify \( \frac{f(x) - f(3)}{x - 3} \).

For problems 6-11, compute the domain of the given function. Express your answer as an inequality and in interval notation.

6. \( f(x) = 5x^2 - 3x + 1 \)
7. \( \frac{3x}{x - 5} \)
8. \( \sqrt{10 - x} \)
9. \( \sqrt{10 - x^2} \)
10. \( \sqrt[3]{10 - x} \)
11. \( \frac{1}{x} - \frac{1}{x - 2} \)

12. Write the area \( A \) of a square as a function of its perimeter \( P \).

For problems 13-14, compute \( f \circ g \), \( g \circ f \), and \( f \circ f \)

13. \( f(x) = x^2 \), \( g(x) = x - 1 \)
14. \( f(x) = \sqrt{x + 1} \), \( g(x) = x^2 + 5 \)
15. Let \( f(x) \) and \( g(x) \) defined as in the table below:

\[
\begin{array}{c|cccc}
  x & 1 & 2 & 3 & 4 \\
  f(x) & 4 & 1 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|cccc}
  x & 1 & 2 & 3 & 4 \\
  g(x) & 2 & 4 & 1 & 3 \\
\end{array}
\]

Compute each of the following:

(a) \((f \circ g)(4)\)
(b) \(g(f(4))\)
(c) \(f(g(1))\)
(d) \((g \circ g)(4)\)

For problems 16-18, compute \(f^{-1}\). Also state the domain and range of \(f^{-1}\) in interval notation.

16. \( f(x) = 3x + 1 \)

17. \( f(x) = \sqrt[3]{x - 1} \)

18. \( f(x) = x^2 + 9, x \geq 0 \)

19. Determine whether the following functions have inverses.
Exponents & Logs

QUICK REVIEW

PRACTICE PROBLEMS

1. Simplify:

(a) \( \left( \frac{49}{100} \right)^{-3/2} \)

(b) \( 5a^{2/3} \cdot 4a^{3/2} \)

(c) \( (4a^{5/3})^{3/2} \)

(d) \( e^{\ln 3} \)

(e) \( \ln 1 \)

(f) \( e^{3 \ln x} \)

(g) \( \log_4 16 \)

(h) \( \log_{1/2} 8 \)

2. For each of the following, use properties of logarithms to expand (as much as possible) the given expression as a sum, difference, and or constant multiple of logarithms. (Assume that all variables are positive)

(a) \( \log_5 (5x^2 \sqrt{y}) \)

(b) \( \ln \sqrt[4]{x^3(x^2 + 3)} \)

3. Solve for \( x \). Where appropriate, you may leave your answer in logarithmic form.

(a) \( e^x + 5 = 60 \)

(b) \( 3^{x-5} - 4 = 11 \)

(c) \( 2 \log_5 (3x) = 4 \)

(d) \( \log_3 x + \log_3 (x - 8) = 2 \)

(e) \( \frac{1 + \ln x}{2} = 0 \)

4. The equation \( Q(t) = 30e^{-4t} \) gives the mass (in grams) of a radioactive element that will remain from some initial quantity after \( t \) hours of radioactive decay.

(a) How many grams were present initially?

(b) How long will it take for 40% of the element to decay? (You may leave your answer in logarithmic form.)
Trigonometry

QUICK REVIEW
PRACTICE PROBLEMS

1. Convert the following angles from degrees to radians.

Graphs of Elementary Functions

PRACTICE PROBLEMS:

1. For each of the following, (i) Sketch the given function, labeling all intersections with the coordinate axes (For trigonometric functions, just sketch one period); (ii) Label all horizontal/vertical asymptotes, if any exist; (iii) Express the domain and range in interval notation.

(a) The Identity Function, \( y = x \)
(b) The Square Function, \( y = x^2 \)
(c) The Cube Function, \( y = x^3 \)
(d) The Absolute Value function, \( y = |x| \).
   (Also, express this function as a ”Piecewise Function”.)
(e) The Square-Root Function, \( y = \sqrt{x} \)
(f) The Cube-Root Function, \( y = \sqrt[3]{x} \)
(g) The Natural Log Function, \( y = \ln x \)
(h) The Exponential Function, \( y = e^x \)
(i) The Sine Function, \( y = \sin x \)
(j) The Cosine Function, \( y = \cos x \)
(k) The Tangent Function, \( y = \tan x \)
(l) The Cotangent Function, \( y = \cot x \)
(m) The Secant Function, \( y = \sec x \)
(n) The Cosecant Function, \( y = \csc x \)
(o) The Inverse Tangent Function, \( y = \tan^{-1} x = \arctan x \)