MATH 200 FINAL EXAM SPRING 2010-2011
June 8, 2011

Name:  
Section:  

ONLY THE CORRECT ANSWER AND ALL WORK USED TO REACH IT WILL EARN FULL CREDIT.

Simplify all answers as much as possible unless explicitly stated otherwise.
This is a closed-book, closed-notes exam. No electronic devices are allowed.

IF YOUR SECTION NUMBER IS MISSING OR INCORRECT, 5 POINTS WILL BE DEDUCTED FROM YOUR SCORE.

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Total: ____________
1. (8 points) Find the equation of the plane that passes through the point \((-4, 7, 12)\) and is perpendicular to the line \(x = 11 - 3t, y = 7 - 10t, z = -2 + 5t\).

\((-3, -10, 5)\) is parallel to the line and thus normal to the plane.

Plane:

\[-3(x + 4) - 10(y - 7) + 5(z - 12) = 0\]

\[-3x - 12 - 10y + 70 + 5z - 60 = 0\]

\[-3x - 10y + 5z = 2\]
1. (8 points) Find the equation of the plane that passes through the point \((2,6,-9)\) and is perpendicular to the line \(x = 15 - 8t, y = -3 + 4t, z = 5 - t\).

\((-8, 4, -1)\) is parallel to the line and thus normal to the plane.

Plane:

\[-8(x-2) + 4(y-6) - 1(z+9) = 0\]

\[-8x + 16 + 4y - 24 - z - 9 = 0\]

\[-8x + 4y - z = 17\]
2. (10 points) Use vectors to find the acute angle formed by any two diagonals of a cube of side length 1. You may leave your answer in the form of an inverse trigonometric function. Recall that a diagonal of a cube runs from one corner of the cube to its opposite corner, e.g. from the lower left corner on the back of the cube to the upper right corner on the front of the cube (see figure below).

\[ \overrightarrow{PQ} = <1,1,1>, \quad \overrightarrow{RS} = <1,1,-1> \]

are diagonals of the cube

\[
\cos \theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{RS}}{||\overrightarrow{PQ}|| ||\overrightarrow{RS}||} = \frac{1 + 1 - 1}{\sqrt{3} \sqrt{3}} = \frac{1}{3}
\]

So \[ \theta = \arccos \frac{1}{3} \]
3. (10 points) Identify all critical points of the given function and classify each critical point as a relative maximum, relative minimum, or saddle point.

\[ f(x, y) = x^3 - 3xy + \frac{1}{2}y^2 \]

\[ f_x = 3x^2 - 3y = 0 \quad \Rightarrow \quad y = x^2 \]

\[ f_y = -3x + y = 0 \quad \Rightarrow \quad y = 3x \]

\[ x^2 = 3x \quad \Rightarrow \quad x(x - 3) = 0 \]

\[ x = 0, x = 3 \]

Critical points: \((0, 0), (3, 9)\)

\[ f_{xx} = 6x, \; f_{yy} = 1, \; f_{xy} = -3 \]

\[ D(x, y) = (6x)(1) - (-3)^2 = 6x - 9 \]

\[ D(0, 0) = -9 < 0 \]

So there is a saddle point at \((0, 0)\)

\[ D(3, 9) = 18 - 9 > 0, \; f_{xx}(3, 9) = 18 > 0 \]

So there is a relative minimum at \((3, 9)\)
4. (10 points) Find the tangent line to the curve of intersection of the two surfaces 
\( x^2y + 2yz - y^2 = -7 \) and \( 2x - 3y + 3z = 10 \) at \((2, -1, 1)\). Your answer may be a vector equation or a set of parametric equations.

Let \( f(x, y, z) = x^2y + 2yz - y^2 \)

\[ \nabla f(x, y, z) = \langle 2xy, x^2 + 2z - 2y, 2y \rangle \]

\[ \nabla f(2, -1, 1) = \langle -4, 8, -2 \rangle \]

Let \( g(x, y, z) = 2x - 3y + 3z \)

\[ \nabla g(x, y, z) = \langle 2, -3, 3 \rangle \]

\[ \nabla f(2, -1, 1) \times \nabla g(2, -1, 1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 8 & -2 \\ -4 & 8 & -2 \end{vmatrix} = 2(24 - 6) - 8(-12 + 4) + 2(12 - 16) = \langle 18, 8, -4 \rangle \parallel \text{tangent line} \]

Tangent line:
\[
\begin{align*}
  x &= 2 + 18t \\
  y &= -1 + 8t \\
  z &= 1 - 4t
\end{align*}
\]
4. (10 points) Find the tangent line to the curve of intersection of the two surfaces 
\( x + 5y - 2z = 20 \) and \( xz^2 - x^2 + 3xy = 12 \) at \((1, 3, -2)\). Your answer may be a vector equation or a set of parametric equations.

Let \( f(x, y, z) = x + 5y - 2z \)

\[ \nabla f(1, 3, -2) = \langle 1, 5, -2 \rangle \]

Let \( g(x, y, z) = xz^2 - x^2 + 3xy \)

\[ \nabla g(x, y, z) = \langle z^2 - 2x + 3y, 3x, 2xz \rangle \]

\[ \nabla g(1, 3, -2) = \langle 11, 3, -4 \rangle \]

\[ \nabla f(1, 3, -2) \times \nabla g(1, 3, -2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{i} & 5 & -2 \\ 11 & 3 & -4 \end{vmatrix} = \mathbf{i}(-20 + 6) - \mathbf{j}(-4 + 22) + \mathbf{k}(3 - 55) \]

\[ = \langle -14, -18, -52 \rangle \parallel \text{tangent line} \]

Tangent line:

\[ \begin{align*}
  x &= 1 - 14t \\
  y &= 3 - 18t \\
  z &= -2 - 52t
\end{align*} \]
5. (10 points) Evaluate the double integral \( \iint_{R} \sin(y^3) \, dA \) where \( R \) is the region bounded by 

\( y = \sqrt{x}, \quad y = 2, \) and \( x = 0. \)

\[
\begin{align*}
\int_{0}^{2} \int_{0}^{y} \sin(y^3) \, dx \, dy \\
= \int_{0}^{2} y^2 \sin(y^3) \, dy \\
= \frac{1}{3} \int_{0}^{8} \sin(u) \, du \\
= -\frac{1}{3} \cos(u) \bigg|_{0}^{8} \\
= -\frac{1}{3} \left( \cos 8 - 1 \right)
\end{align*}
\]
6. (10 points) Evaluate the iterated integral by converting to polar coordinates.

\[
\int_{0}^{\sqrt{4-x^2}} \int_{x}^{\sqrt{4-x^2}} 4xy \, dy \, dx
\]

\[y = \sqrt{4-x^2} \iff r = 2\]
\[y = x \iff \theta = \frac{\pi}{4}\]

\[
\frac{\pi}{4} \int_{0}^{2} 4r^3 \cos \theta \sin \theta \, dr \, d\theta = \frac{\pi}{4} \int_{0}^{2} 4r^3 \cos \theta \sin \theta \, dr \, d\theta
\]

\[
= \frac{\pi}{4} \int_{0}^{\frac{\pi}{2}} 16 \cos \theta \sin^2 \theta \, d\theta
\]
\[u = \sin \theta \]
\[du = \cos \theta \, d\theta\]

\[
= 16 \int_{0}^{1} u \, du = 8u^2 \bigg|_{0}^{1} = 8 \left(1 - \frac{1}{2}\right) = 4
\]
7. (7 points) On the graph below shade the regions that are inside $r = 3$ and outside $r = 3\cos 2\theta$. In polar coordinates, give an iterated integral (or integrals) that represent(s) the total area of these regions. DO NOT EVALUATE THE INTEGRAL(S).

![Graph showing regions shaded and an iterated integral expression]

$$\int_{\pi/4}^{\pi} \int_0^{3\cos 2\theta} r \, dr \, d\theta$$

There are other correct answers.
8. (10 points) Find the surface area of the portion of the surface \( z = x^2 + 3y \) that is above the triangular region with vertices \((0,0),(2,0),\) and \((2,2)\).

\[
\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 3
\]

\[
\iint_R \sqrt{\left(2x\right)^2 + 3^2 + 1} \, dA
\]

\[
= \int_0^2 \int_0^{4x^2+10} \sqrt{4x^2+10} \, dy \, dx = \int_0^2 x \sqrt{4x^2+10} \, dx \quad \text{with} \quad u = 4x^2 + 10 \quad \text{and} \quad du = 8x \, dx
\]

\[
= \frac{1}{8} \int_{10}^{26} \sqrt{u} \, du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \bigg|_{10}^{26} = \frac{1}{12} \left( 26\sqrt{26} - 10\sqrt{10} \right)
\]
8. (10 points) Find the surface area of the portion of the surface $z = 2x + y^2$ that is above the triangular region with vertices $(0,0),(0,3),$ and $(3,3)$.

$$\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 2y$$

$$\iint_R \sqrt{(2)^2 + (2y)^2 + 1} \, dA$$

$$= \int_0^3 \int_0^y \sqrt{4y^2 + 5} \, dx \, dy = \int_0^3 y \sqrt{4y^2 + 5} \, dy \quad \text{and} \quad u = 4y^2 + 5$$

$$du = 8y \, dy$$

$$= \frac{1}{8} \int_5^{41} \sqrt{u} \, du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \bigg|_5^{41} = \frac{1}{12} \left( 41 \sqrt{41} - 5 \sqrt{5} \right)$$
9. (5 points each) Consider the solid bounded above by the sphere \( x^2 + y^2 + z^2 = 16 \) and bounded below by the cone \( z = \frac{\sqrt{x^2 + y^2}}{3} \).

a. In rectangular coordinates give an iterated integral that represents the volume of the solid.

b. In cylindrical coordinates give an iterated integral that represents the volume of the solid.

c. In spherical coordinates give an iterated integral that represents the volume of the solid.

DO NOT EVALUATE ANY OF THE INTEGRALS.

\[
\begin{align*}
\text{(a)} & \quad \int_{-2\sqrt{3}}^{2\sqrt{3}} \int_{-\sqrt{12-x^2}}^{\sqrt{12-x^2}} \int_{\sqrt{16-x^2-y^2}}^{\sqrt{x^2+y^2}} 1 \, dz \, dy \, dx \\
\text{(b)} & \quad \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{3}} \int_{0}^{\frac{\sqrt{16-r^2}}{\sqrt{3}}} r \, dr \, dz \, d\theta \\
\text{(c)} & \quad \int_{0}^{2\pi} \int_{\frac{\pi}{3}}^{4} \int_{0}^{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\end{align*}
\]
10. (10 points) Use the transformation \( u = x + y, v = x - y \) to find \( \iint_R \sin(x + y) \cos(x - y) \, dA \)

where \( R \) is the triangular region with vertices \((0,0), (2,0), \) and \((1,1)\).

Hint: If \( u = x + y, v = x - y \), then \( x = \frac{1}{2} (u + v), y = \frac{1}{2} (u - v) \).

\[ \frac{\partial (x,y)}{\partial (u,v)} = \left| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right| = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \]

1. \( y = 0 \Rightarrow 0 = \frac{1}{2} (u - v) \Rightarrow u = v 
2. \( y = x \Rightarrow x - y = 0 \Rightarrow v = 0 
3. \( y = 2 - x \Rightarrow x + y = 2 \Rightarrow u = 2 

\frac{1}{2} \iint_S \sin u \cos v \, dA = \frac{1}{2} \int_0^2 \sin u \sin v \, dv \, du = \frac{1}{2} \int_0^2 \sin u \sin v \bigg|_0^u \, du = \frac{1}{2} \int_0^2 \sin^2 u \, du = \frac{1}{4} \int_0^2 (1 - \cos 2u) \, du = \frac{1}{4} \left[ u \bigg|_0^2 - \frac{1}{2} \sin 2u \bigg|_0^2 \right] = \frac{1}{4} \left( 2 - \frac{1}{2} \sin 4 \right)

Note: \( \frac{1}{2} \iint_S \sin u \cos v \, du \, dv = \ldots \) is also OK
Space for extra work