Tangent Planes & Normal Lines

SUGGESTED REFERENCE MATERIAL:
As you work through the problems listed below, you should reference Chapter 13.7 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

• Be able to compute an equation of the tangent plane at a point on the surface \( z = f(x, y) \).
• Given an implicitly defined level surface \( F(x, y, z) = k \), be able to compute an equation of the tangent plane at a point on the surface.
• Know how to compute the parametric equations (or vector equation) for the normal line to a surface at a specified point.
• Be able to use gradients to find tangent lines to the intersection curve of two surfaces. And, be able to find (acute) angles between tangent planes and other planes.

PRACTICE PROBLEMS:
For problems 1-4, find two unit vectors which are normal to the given surface \( S \) at the specified point \( P \).

1. \( S: 2x - y + z = -7; \ P(-1, 2, -3) \)
   \[ \vec{n}_{1,2} = \pm \left\langle \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle \]

2. \( S: x^2 - 3y + z^2 = 11; \ P(-1, -2, 2) \)
   \[ \vec{n}_{1,2} = \pm \left\langle -\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle \]

3. \( S: z = y^4; \ P(3, -1, 1) \)
   \[ \vec{n}_{1,2} = \pm \left\langle 0, -\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle \]

4. \( S: z = 2 - x^2 \cos(xy); \ P\left(-1, \frac{\pi}{2}, 2\right) \)
   \[ \vec{n}_{1,2} = \pm \frac{2}{\sqrt{\pi^2 + 8}} \left\langle -\frac{\pi}{2}, 1, -1 \right\rangle \]
For problems 5-9, compute equations of the tangent plane and the normal line to the given surface at the indicated point.

5. \( S : \ln (x + y + z) = 2; \ P(-1, e^2, 1) \)

\[
x + y + z = e^2; \quad \vec{\ell} (t) = \langle -1, e^2, 1 \rangle + t \langle 1, 1, 1 \rangle
\]

6. \( S : x^2 + y^2 + z^2 = 1; \ P\left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right) \)

\[
x + y + z = \sqrt{3}; \quad \vec{\ell} (t) = \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle + t \langle 1, 1, 1 \rangle
\]

7. \( S : z = \arcsin \left( \frac{x}{y} \right); \ P\left( -1, -\sqrt{2}, \frac{\pi}{4} \right) \)

\[
x + \frac{\sqrt{2}}{2} y - z = -\frac{\pi}{4}; \quad \vec{\ell} (t) = \langle -1, -\sqrt{2}, \frac{\pi}{4} \rangle + t \langle -1, -\frac{\sqrt{2}}{2}, -1 \rangle
\]

8. \( S : x^2 - xy + z^2 = 9; \ P(2, 2, 3) \)

\[
x - y + 3z = 9; \quad \vec{\ell} (t) = \langle 2, 2, 3 \rangle + t \langle 1, -1, 3 \rangle
\]

9. \( S : z = x \cos (x + y); \ P\left( \frac{\pi}{2}, \frac{\pi}{3}, -\frac{\sqrt{3}\pi}{4} \right) \)

\[
(\pi + 2\sqrt{3}) \left( x - \frac{\pi}{2} \right) + \pi \left( y - \frac{\pi}{3} \right) + 4 \left( z + \frac{\sqrt{3}\pi}{4} \right) = 0
\]

\[
\vec{\ell} (t) = \left\langle \frac{\pi}{2}, \frac{\pi}{3}, -\frac{\sqrt{3}\pi}{4} \right\rangle + t \left\langle \pi + 2\sqrt{3}, \pi, 4 \right\rangle
\]

10. Consider the surfaces \( S_1 : x^2 + y^2 = 25 \) and \( S_2 : z = 2 - x \)

(a) Find an equation of the tangent line to the curve of intersection of \( S_1 \) and \( S_2 \) at the point \( (3, 4, -1) \).

\[
\vec{\ell} (t) = \langle 3, 4, -1 \rangle + t \langle -4, 3, 4 \rangle
\]

(b) Find the acute angle between the planes which are tangent to the surfaces \( S_1 \) and \( S_2 \) at the point \( (3, 4, -1) \).

\[
\pi - \cos^{-1} \left( \frac{-3}{5\sqrt{2}} \right)
\]
11. Consider the surfaces $S_1 : z = x^2 - y^2$ and $S_2 : y^2 + z^2 = 10$

(a) Find an equation of the tangent line to the curve of intersection of $S_1$ and $S_2$ at the point $(2, 1, 3)$.

\[ \overrightarrow{\ell}(t) = (2, 1, 3) + t(5, 12, -4) \]

(b) Find the acute angle between the planes which are tangent to the surfaces $S_1$ and $S_2$ at the point $(2, 1, 3)$.

\[ \pi - \cos^{-1} \left( \frac{-10}{\sqrt{21}\sqrt{40}} \right) \]

12. Find all points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 72$ where the tangent plane is parallel to the plane $4x + 4y + 12z = 3$.

$$(4, 2, 4) \text{ and } (-4, -2, -4)$$

13. Find all points on the hyperboloid of 1 sheet $x^2 + y^2 - z^2 = 9$ where the normal line is parallel to the line which contains points $A(1, 2, 3)$ and $B(7, 6, 5)$.

\[
\left( \frac{3\sqrt{3}}{2}, \sqrt{3}, -\frac{\sqrt{3}}{2} \right) \text{ and } \left( -\frac{3\sqrt{3}}{2}, -\sqrt{3}, \frac{\sqrt{3}}{2} \right)
\]

14. Two surfaces are called orthogonal at a point of intersection if their normal lines are perpendicular at that point. Show that the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z^2 = x^2 + y^2$ are orthogonal at all points of intersection. (HINT: Assume that the surfaces intersect at the arbitrary point $(x_0, y_0, z_0)$.)

Suppose that $S_1 : x^2 + y^2 + z^2 = 1$ and $S_2 : z^2 = x^2 + y^2$ intersect at $(x_0, y_0, z_0)$. We will find a normal vector to each surface at the point $P_0$. To do this, let $F(x, y, z) = x^2 + y^2 + z^2$ and $G(x, y, z) = x^2 + y^2 - z^2$. Notice that $S_1$ is the level surface $F(x, y, z) = 1$ and $S_2$ is the level surface $G(x, y, z) = 0$. So, $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 2z_0 \rangle$ and $\nabla G(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle$ are normal to $S_1$ and $S_2$, respectively, at the point $P_0$. And, as a result, these vectors are parallel to the normal lines to $S_1$ and $S_2$ at $P_0$.

Showing that the surfaces are orthogonal is equivalent to showing that $\nabla F(x_0, y_0, z_0) \perp \nabla G(x_0, y_0, z_0)$; i.e, $\nabla F(x_0, y_0, z_0) \cdot \nabla G(x_0, y_0, z_0) = 0$.

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\[ \nabla F(x_0, y_0, z_0) \cdot \nabla G(x_0, y_0, z_0) = \langle 2, 2, 2 \rangle \cdot \langle 2x_0, 2y_0, -2z_0 \rangle \\
= 4x_0^2 + 4y_0^2 - 4z_0^2 \\
= 4(x_0^2 + y_0^2 - z_0^2) \]

Since \((x_0, y_0, z_0)\) is a point of intersection of \(S_1\) and \(S_2\), it must satisfy both equations. In particular, since it satisfies the equation for \(S_2\), we have \(x_0^2 + y_0^2 = z_0^2\). Using this fact, we see that

\[ \nabla F(x_0, y_0, z_0) \cdot \nabla G(x_0, y_0, z_0) = 4(x_0^2 + y_0^2 - z_0^2) \\
= 4(z_0^2 - z_0^2) \\
= 0 \]

As a result, the surfaces are orthogonal to one another at the point of intersection, \((x_0, y_0, z_0)\).

15. Show that every plane which is tangent to the cone \(z^2 = x^2 + y^2\) must pass through the origin. (HINT: Compute the equation of the plane which is tangent to the surface at the point \(P_0(x_0, y_0, z_0)\) and see what happens.)

Let \(F(x, y, z) = x^2 + y^2 - z^2\). The given surface is the level surface \(F(x, y, z) = 0\); so, \(\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle\) is normal to the given surface at the point \((x_0, y_0, z_0)\). Thus, an equation of the plane which is tangent to the given surface at the point \((x_0, y_0, z_0)\) is \(2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0\); i.e., \(x_0 x + y_0 y + z_0 z - x_0^2 - y_0^2 + z_0^2 = 0\).

Now, since \((x_0, y_0, z_0)\) is the point of tangency, it must also be a point on the surface. Thus, \(x_0^2 + y_0^2 = z_0^2 \Rightarrow -x_0^2 - y_0^2 = -z_0^2\). Using this fact, the equation of the tangent plane can be written as:

\[ x_0 x + y_0 y + z_0 z - x_0^2 - y_0^2 + z_0^2 = 0 \\
x_0 x + y_0 y + z_0 z - z_0^2 + z_0^2 = 0 \\
x_0 x + y_0 y + z_0 z = 0 \]

And, \((0, 0, 0)\) satisfies this equation. Thus, since \((x_0, y_0, z_0)\) was an arbitrary point on the surface and its tangent plane passes through the origin, we have that all tangent planes to the surface must pass through the origin.