Math 200 - Exam 2 - 5/22/2015

Name: SOLUTIONS

Section

<table>
<thead>
<tr>
<th>Section</th>
<th>Class Times</th>
<th>Instructor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9:00 AM - 9:50 AM</td>
<td>Papadopoulos, Dimitrios</td>
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<tr>
<td>2</td>
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<td>Lee, Hwan Yong</td>
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<td>4</td>
<td>4:00 PM - 4:50 PM</td>
<td>Aran, Jason</td>
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<td>6</td>
<td>9:00 AM - 9:50 AM</td>
<td>Lee, Hwan Yong</td>
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<tr>
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<td>Swartz, Kenneth</td>
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<tr>
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<td>Aran, Jason</td>
</tr>
<tr>
<td>16</td>
<td>12:00 PM - 12:50 PM</td>
<td>Zhang, Aljun</td>
</tr>
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<td>17</td>
<td>4:00 PM - 4:50 PM</td>
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</tr>
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<td>1:00 PM - 1:50 PM</td>
<td>Yang, Dennis</td>
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<td>Lee, Hwan Yong</td>
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<td>20</td>
<td>1:00 PM - 1:50 PM</td>
<td>Akin, Myles</td>
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The following rules apply:

- This is a closed-book exam. You may not use any books or notes on this exam.

- For free response questions, you must show all work. Answers without proper justification will not receive full credit. Partial credit will be awarded for significant progress towards the correct answer. Cross off any work that you do not want graded.

- For multiple choice questions, circle the letter of the best answer. Make sure your circles include just one letter. These problems will be marked as correct or incorrect; partial credit will not be awarded for problems in this section.

- You have 50 minutes to complete this exam. When time is called, stop writing immediately and turn in your exam to the nearest proctor.

- You may not use any electronic devices including (but not limited to) calculators, cell phone, or iPods. Using such a device will be considered a violation of the university’s academic integrity policy and, at the very least, will result in a grade of 0 for this exam.

<table>
<thead>
<tr>
<th>Page</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16</td>
<td></td>
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Part I: Free Response

1. (16 points) Identify all critical points of the given function. Then, classify each as the location of a relative maximum, relative minimum, or saddle point.

\[ f(x,y) = 6xy - \frac{3}{2}x^2 - 2y^3 \]

\[
\begin{align*}
\frac{\partial}{\partial x} &= 6y - 3x = 0 \\
\frac{\partial}{\partial y} &= 6x - 6y^2 = 0
\end{align*}
\]

\[ 6(2y) - 6y^2 = 0 \]
\[ 6y(2-y) = 0 \]
\[ y = 0, 2 \]
\[ x = z(0) = 0; \quad x = z(2) = 4 \]

**Critical Points:** \((0,0)\); \((4,2)\)

\[ \frac{\partial^2}{\partial x^2} = -3 \quad \frac{\partial^2}{\partial y^2} = -12y \quad \frac{\partial^2}{\partial x \partial y} = 6 \]

\[ D(0,0) = (-3)(-12(0)) - (6)^2 = -36 < 0 \]

\[ D(4,2) = (-3)(-12(2)) - (6)^2 = 36 > 0 \]

\[ D_{xx}(4,2) = -3 < 0 \]

\[ \therefore \text{HAS A SADDLE POINT AT } (0,0) \text{ AND A RELATIVE MAXIMUM AT } (4,2). \]
2. (18 points) Let \( f(x, y, z) = x^2y^3z^4 \).

(a) Compute the directional derivative of \( f \) at the point \( P(1, 1, 1) \) in the direction of vector \( \mathbf{v} = (1, 3, 2) \).

(b) Compute a unit vector in the direction in which \( f \) increases most rapidly at \( P \).

(c) Find the maximum rate of change of \( f \) at \( P \).

\[
\nabla f = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle
\]

\[
\nabla f(1, 1, 1) = \langle 2, 3, 4 \rangle
\]

\[
\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{\langle 1, 3, 2 \rangle}{\sqrt{1^2 + 3^2 + 2^2}} = \frac{1}{\sqrt{14}} \langle 1, 3, 2 \rangle
\]

\[
D_\mathbf{u} f(1, 1, 1) = \langle 2, 3, 4 \rangle \cdot \langle 1, 3, 2 \rangle \frac{1}{\sqrt{14}} = (2 + 9 + 8) \left(\frac{1}{\sqrt{14}}\right) = \frac{19}{\sqrt{14}}
\]

\[
\frac{\nabla f(1, 1, 1)}{||\nabla f(1, 1, 1)||} = \frac{\langle 2, 3, 4 \rangle}{\sqrt{2^2 + 3^2 + 4^2}} = \frac{\langle 2, 3, 4 \rangle}{\sqrt{29}} = \langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \rangle
\]

(c) Maximum rate of change is \( \sqrt{29} \).
3. (16 points) Let $R$ be the region in the first quadrant bounded by $x = 0$, $y = 2$, and $y = \sqrt{x}$, shown below.

(a) Set up $\int\int_R \sqrt{y^2 + 1} \, dA$ with the order of integration as $dy \, dx$

\[
\int_{\sqrt{x}}^{2} \int_{0}^{\sqrt{x}} \sqrt{y^2 + 1} \, dy \, dx
\]

Intersection:
\[
\begin{align*}
  y &= \sqrt{x} \\
  z &= \frac{1}{\sqrt{x}} \\
  x &= 4
\end{align*}
\]

(b) Set up $\int\int_R \sqrt{y^2 + 1} \, dA$ with the order of integration as $dx \, dy$

\[
\int_{0}^{2} \int_{y^2}^{\sqrt{x}} \sqrt{y^2 + 1} \, dx \, dy
\]

$y = \sqrt{x} \rightarrow x = y^2$
(c) Evaluate \( \iint_R \sqrt{y^3 + 1} \, dA \) by evaluating either your integral from part (a) or part (b).

\[
\begin{align*}
\int_0^2 \int_0^{y^2} \sqrt{y^3 + 1} \, dx \, dy &= \int_0^2 \left[ \sqrt{x} \sqrt{y^3 + 1} \right]_{x=0}^{x=y^2} \, dy \\
&= \int_0^2 y^2 \sqrt{y^3 + 1} \, dy \\
&= \int_0^9 u^{\frac{2}{3}} \, du \quad \text{sub: } u = y^3 + 1, \, du = 3y^2 \, dy \\
&= \frac{1}{3} \left[ u^{\frac{5}{3}} \right]_1^9 \\
&= \frac{1}{3} \left( \frac{2^7}{3} \right) \left( \frac{9}{3} \right) \\
&= \frac{2}{9} \left( 27 - 1 \right) \\
&= \frac{52}{9}
\end{align*}
\]
4. (15 points) Consider the surface \( S : z = x^2 + y^2 \).

(a) Find an equation of the plane which is tangent to \( S \) at \( P(1, 2, 5) \).

(b) Find the acute angle between the tangent plane to \( S \) at \( P(1, 2, 5) \) and the plane \( x + z = 6 \).
(You may leave your answer in terms of an inverse trigonometric function.)

\[
\mathbf{r} = x^2 + y^2 \quad \Rightarrow \quad \mathbf{0} = \frac{x^2 + y^2 - z}{F(x, y, z)}
\]

\[
\nabla F = \langle 2x, 2y, -1 \rangle
\]

\[
\nabla F(1,2,5) = \langle 2, 4, -1 \rangle
\]

TANGENT PLANE: \( 2(x-1) + 4(y-2) - (z-5) = 0 \)

\[
\begin{align*}
\langle 2, 4, -1 \rangle \cdot \langle 1, 0, 1 \rangle &= \cos \theta \\
\| \langle 2, 4, -1 \rangle \| &\| \langle 1, 0, 1 \rangle \| \\
\frac{1}{\sqrt{21} \sqrt{2}} &= \cos \theta \\
\theta &= \cos^{-1} \left( \frac{1}{\sqrt{42}} \right)
\end{align*}
\]
Part II: Multiple Choice

5. (5 points) Consider the level curves of the function \( z = f(x, y) \) shown below. Which of the following best describes the signs of the partial derivatives at the point \( P(x_0, y_0) \)?

- (a) \( f_x(x_0, y_0) > 0 \), \( f_y(x_0, y_0) > 0 \)
- (b) \( f_x(x_0, y_0) > 0 \), \( f_y(x_0, y_0) < 0 \)
- (c) \( f_x(x_0, y_0) < 0 \), \( f_y(x_0, y_0) > 0 \)
- (d) \( f_x(x_0, y_0) < 0 \), \( f_y(x_0, y_0) < 0 \)
- (e) \( f_x(x_0, y_0) = 0 \), \( f_y(x_0, y_0) = 0 \)
6. (5 points) If \( f(x, y) = \ln \left( x^2 + 2y^2 \right) \), what is \( f_{xy}(x, y) \)?

(a) \(-\frac{2x}{(x^2 + 2y^2)^2}\)

\[ f_x = \frac{z}{x} = \frac{2x}{x^2 + 2y^2} \]

\[ f_{xy} = \frac{-z}{(x^2 + 2y^2)^2} \]

(b) \(-\frac{4y}{(x^2 + 2y^2)^2}\)

(c) \(-\frac{8xy}{(x^2 + 2y^2)^2}\)

(d) \frac{2xy}{(x^2 + 2y^2)^2}\)

(e) \frac{4(x^2 - y^2)}{(x^2 + 2y^2)^2}\)

7. (5 points) At what rate is \( z = \frac{x^2}{9} + \frac{y^2}{4} \) changing with respect to \( y \) at the point \( (3, 2, 2) \)?

(a) \( \frac{2}{3} \)

\[ \frac{\partial z}{\partial y} = \frac{y}{2} \]

(b) \( \frac{13}{18} \)

(c) \( 1 \)

\[ \frac{\partial z}{\partial y} \bigg|_{(3, 2, 2)} = 1 \]

(d) \( \frac{5}{3} \)

(e) \( 7 \)
8. (5 points) The figure below shows three level curves of \( z = f(x, y) \). Points \( P, Q, R, S, \) and \( T \) are on the level curve \( f(x, y) = 20 \).

Which of the following has the greatest magnitude?

(a) \( \nabla f(P) \)

(b) \( \nabla f(Q) \)

(c) \( \nabla f(R) \)

(d) \( \nabla f(S) \)

(e) \( \nabla f(T) \)
9. (5 points) Consider the integral \( \int_0^2 \int_{x^2}^{2x} f(x, y) \, dy \, dx \). Switching the order of integration to \( dx \, dy \) yields \( \int_0^y \int_a^b f(x, y) \, dx \, dy \). What are \( a \) and \( b \)?

(a) \( a = y^2 \), \( b = 2y \)

(b) \( a = \frac{y}{2} \), \( b = \sqrt{y} \)

(c) \( a = \frac{y}{2} \), \( b = y \)

(d) \( a = \sqrt{y} \), \( b = \frac{y}{2} \)

(e) Cannot be determined without explicit knowledge of \( f(x, y) \).

10. (5 points) Suppose \( f(x, y) \) is a differentiable function at the point \( (2, 3) \) such that \( f(2, 3) = 1 \), \( f_x(2, 3) = 5 \) and \( f_y(2, 3) = -6 \). Use the local linear approximation of \( f(x, y) \) at \( (2, 3) \) to estimate \( f(1, 4) \).

(a) \( L(1, 4) = 2 \)

(b) \( L(1, 4) = 9 \)

(c) \( L(1, 4) = 13 \)

(d) \( L(1, 4) = -10 \)

(e) \( L(1, 4) = -26 \)
11. (5 points) Consider the region \( R = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\} \), shown below.

![Image](image.png)

At which point on this region does \( f(x, y) = x - y \) have an absolute (global) maximum?

(a) (1, 1)

(b) (1, -1)

(c) (-1, 1)

(d) (-1, -1)

(c) The function does not have an absolute maximum on this region since there are no critical points.