Chapter 13.4: Local Linear Approximations; Differentials

Review:

Tangent line to \( y = f(x) \) at \( x_0 \)

Equation: \( y - y_0 = m(x - x_0) \)

\[
\begin{align*}
\Delta x & \\
L(x) & = f(x_0) + f'(x_0)(x - x_0)
\end{align*}
\]

\( L(x) \) is the local linear approximation to \( f \) at \( x_0 \)

So when \( x \) is near \( x_0 \), \( f(x) \approx L(x) \)

E.g. \( 3\sqrt{x} \approx \frac{1}{3} x + \frac{2}{3} \) near \( x_0 = 1 \)

(See Calc I notes if you need more details)

So \( 3\sqrt{1.1} \approx \frac{1}{3}(1.1) + \frac{2}{3} = 1.0333 \) Calculator: \( 3\sqrt{1.1} \approx 1.03228 \)
New stuff:

\[ L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

is the \underline{local linear approximation} to \( f(x, y) \) at \((x_0, y_0)\)

[Note: We will see that \( z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \)

is the equation of the tangent plane to \( f(x, y) \) at \((x_0, y_0)\)]

So when \((x, y)\) is near \((x_0, y_0)\), \( f(x, y) \approx L(x, y) \)
Example: Approximate $\sqrt{(3.04)^2 + (3.98)^2}$

So $f(x, y) = \sqrt{x^2 + y^2}$, $(x_0, y_0) = (3, 4)$

$L(x_1, y_1) = f(3, 4) + f_x(3, 4)(x-3) + f_y(3, 4)(y-4)$

Now $f(3, 4) = \sqrt{3^2 + 4^2} = 5$

$f_x = \frac{1}{2\sqrt{x^2 + y^2}} (2x) = \frac{x}{\sqrt{x^2 + y^2}} \Rightarrow f_x(3, 4) = \frac{3}{5}$

$f_y = \frac{1}{2\sqrt{x^2 + y^2}} (2y) = \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow f_y(3, 4) = \frac{4}{5}$

So $L(x_1, y_1) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$

$L(3.04, 3.98) = 5 + \frac{3}{5}(0.04) + \frac{4}{5}(-0.02) = 5.008$

Calculator: $\sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$
Application: Error Approximation (Differentials)

Example: The legs of a right triangle are measured to be 3 cm and 4 cm with a maximum error of 0.05 cm in each measurement. Approximate the maximum possible error in the area of the triangle.

So \( z = f(x, y) = \frac{1}{2}xy \)

\( (x_0, y_0) = (3, 4) \quad |\Delta x| \leq 0.05 \quad |\Delta y| \leq 0.05 \)

[\( \Delta x \) and \( \Delta y \) are the measurement errors of \( x \) and \( y \)]

Strategy: We will approximate the propagation error

\( \Delta z \): the actual change in height of the surface \( f(x, y) \)

when we move from \( (x_0, y_0) \) to \( (x_0 + \Delta x, y_0 + \Delta y) \)

with the differential

\( dz \): the change in height of the tangent plane \( L(x_0, y_0) \)

when we move from \( (x_0, y_0) \) to \( (x_0 + \Delta x, y_0 + \Delta y) \)
\[ \frac{dz}{dx} = L(x, y) - f(x, y_0) \]

\[ dz = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

\[ \Delta x = dx \quad \Delta y = dy \]

So \[ dz = f_x(x, y) \, dx + f_y(x, y) \, dy \]

\text{total differential}

Back to example:

\[ z = f(x, y) = \frac{1}{2} xy \]

\[ |\Delta x| \leq 0.05 \Rightarrow |dx| \leq 0.05 \]

\[ |\Delta y| \leq 0.05 \Rightarrow |dy| \leq 0.05 \]

\[ f_x(x, y) = \frac{1}{2} y \quad \Rightarrow \quad f_x(3, 4) = \frac{1}{2} (4) = 2 \]

\[ f_y(x, y) = \frac{1}{2} x \quad \Rightarrow \quad f_y(3, 4) = \frac{3}{2} \]
So,

\[ dz = f_x(3,4)\, dx + f_y(3,4)\, dy \]

\[ |dz| = 2(0.05) + \frac{3}{2}(0.05) = 0.175 \]

Approximate maximum possible error in area is 0.175 cm².

We can also find the approximate maximum percentage errors in the legs and area of the triangle,

e.g. \[ |\Delta x| = |\frac{dx}{x}| \leq \frac{0.05}{3} = 0.0167 = 1.67\% \]

\[ |\Delta y| = |\frac{dy}{y}| \leq \frac{0.05}{4} = 0.0125 = 1.25\% \]

So,

\[ \frac{dz}{z} = \frac{f_x(x,y)\, dx + f_y(x,y)\, dy}{z} = \frac{\frac{1}{2}y\, dx + \frac{1}{2}x\, dy}{\frac{1}{2}xy} = \frac{dx}{x} + \frac{dy}{y} \]

\[ \Rightarrow |\frac{dz}{z}| \leq 1.67\% + 1.25\% = 2.92\% \]
Note: All of the above can be applied to functions of three variables, e.g., \( W = f(x, y, z) \)

\[
L(x_0, y_0, z_0) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x-x_0) + f_y(x_0, y_0, z_0)(y-y_0) + f_z(x_0, y_0, z_0)(z-z_0)
\]

is the local linear approximation to \( f(x, y, z) \) at \( (x_0, y_0, z_0) \).

[So if \((x_0, y_0, z_0)\) stays close to \((x, y, z)\), then \( f(x, y, z) \approx L(x, y, z) \).]

\[
dW = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz
\]

is the total differential.