Exam 2 - MATH 200: Solutions

Problem 1. (a) Find the gradients of the functions \( f(x,y,z) = 2x^2 + 3y^2 + z^2 \) and \( g(x,y,z) = x^2 + y^2 + z^2 - 6x - 8y - 8z + 24 \);

(b) Show that the ellipsoid \( 2x^2 + 3y^2 + z^2 = 9 \) and the sphere \( x^2 + y^2 + z^2 - 6x - 8y - 8z + 24 = 0 \)

have a common tangent plane at the point \((1,1,2)\).

Solution. (a) The gradients are \( \nabla f(x,y,z) = 4x \mathbf{i} + 6y \mathbf{j} + 2z \mathbf{k} \) and \( \nabla g(x,y,z) = (2x-6) \mathbf{i} + (2y-8) \mathbf{j} + (2z-8) \mathbf{k} \).

(b) We have \( \nabla f(1,1,2) = 4 \mathbf{i} + 6 \mathbf{j} + 4 \mathbf{k} \). Then the equation of the tangent plane to the ellipsoid \( 2x^2 + 3y^2 + z^2 = 9 \) (which is a level surface for the function \( f(x,y,z) \)) is \( 4(x-1) + 6(y-1) + 4(z-1) = 0 \). On the other hand, \( \nabla g(1,1,2) = -4 \mathbf{i} - 6 \mathbf{j} - 4 \mathbf{k} \). Thus the equation of the tangent plane to the sphere \( x^2 + y^2 + z^2 - 6x - 8y - 8z + 24 = 0 \) (which is a level surface for the function \( g(x,y,z) \)) is \( -4(x-1) - 6(y-1) - 4(z-1) = 0 \). Multiplying both sides of this equation by \(-1\), we obtain \( 4(x-1) + 6(y-1) + 4(z-1) = 0 \). Hence, the ellipsoid and the sphere have the same tangent plane at the point \((1,1,2)\).

Problem 2. Locate all relative maxima, minima, and saddle points for \( f(x,y) = y - \sin(xy) \).

Solution. We have \( f_x(x,y) = -y \cos(xy) \) and \( f_y(x,y) = 1 - x \cos(xy) \). Critical points of \( f(x,y) \) must satisfy the equations \(-y \cos(xy) = 0 \) and \( 1 - x \cos(xy) = 0 \). For the first equation, either \( y = 0 \) or \( \cos(xy) = 0 \). If \( \cos(xy) = 0 \) then the equation \( 1 - x \cos(xy) = 0 \) becomes \( 1 = 0 \), a contradiction. Therefore, we necessarily have \( y = 0 \). In this case, \( 0 = 1 - x \cos(0) = 1 - x \), i.e., \( x = 1 \). Hence, we obtain a single critical point \((1,0)\). Next we apply the second derivatives test to determine the type of this critical point. We have \( f_{xx}(x,y) = (-y \cos(xy))_x = y^2 \sin(xy) \), \( f_{yy}(x,y) = (1 - x \cos(xy))_y = x^2 \sin(xy) \), \( f_{xy}(x,y) = f_{yx}(x,y) = (-y \cos(xy))_y = -\cos(xy) + xy \sin(xy) \). At \((1,0)\), we have \( f_{xx}(1,0) = 0 \), \( f_{yy}(1,0) = 0 \), \( f_{xy}(1,0) = f_{yx}(1,0) = -1 \), and \( D = f_{xx}(1,0)f_{yy}(1,0) - f_{xy}(1,0)^2 = 0 \cdot 0 - (-1)^2 = -1 < 0 \). We conclude that \((1,0)\) is a saddle point for \( f(x,y) = y - \sin(xy) \).
Problem 3. Find the point on the plane $2x + y - z = 5$ that is closest to the point $(1, 1, 1)$. (Use Lagrange’s multiplier method.)

Solution. In this problem, we have to minimize the distance (or equivalently, minimize the square of the distance) from a point $(x, y, z)$ on the plane $2x + y - z = 5$ to the point $(1, 1, 1)$, i.e., minimize the quantity $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ subject to the constraint $g(x, y, z) = 2x + y - z - 5 = 0$. The Lagrange equation $\nabla f = \lambda \nabla g$ is equivalent to the three scalar equations

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z,$$

which become

$$2(x - 1) = \lambda \cdot 2, \quad 2(y - 1) = \lambda \cdot 1, \quad 2(z - 1) = \lambda \cdot (-1).$$

Expressing $\lambda$ from these equations we obtain $\lambda = x - 1 = 2y - 2 = 2 - 2z$, hence $x = 2y - 1$, $z = 2 - y$. Now plugging these into the constraint equation we obtain $2(2y - 1) + y - (2 - y) = 0$, or equivalently, $6y - 4 = 0$. We obtain $y = \frac{2}{3}$, $x = 2 \cdot \frac{2}{3} - 1 = 2$, $z = 2 - \frac{2}{3} = \frac{4}{3}$. Therefore, the closest point on the plane $2x + y - z = 5$ to the point $(1, 1, 1)$ is $(2, \frac{2}{3}, \frac{4}{3})$.

Problem 4. Evaluate $\int_{0}^{\pi/2} \int_{0}^{x^2} x \sin(y) \, dy \, dx$.

Solution.

$$\int_{0}^{\pi/2} \int_{0}^{x^2} x \sin(y) \, dy \, dx = \int_{0}^{\pi/2} \left. (-x \cos(y)) \right|_{y=0}^{y=x^2} \, dx$$

$$= \int_{0}^{\pi/2} (-x \cos(x^2) + x \cos 0) \, dx = \int_{0}^{\pi/2} (-x \cos(x^2) + x) \, dx$$

$$= \int_{0}^{\pi/2} x \, dx - \int_{0}^{\pi/2} x \cos(x^2) \, dx.$$

We have

$$\int_{0}^{\pi/2} x \, dx = \frac{x^2}{2} \bigg|_{0}^{\pi/2} = \frac{\pi^2}{8},$$

$$\int_{0}^{\pi/2} x \cos(x^2) \, dx = \left[ u = x^2, \quad du = 2x \, dx \right] \bigg|_{x=0}^{x=\pi/2} = \left[ u = 0 \rightarrow u = \pi^2 / 4 \right]$$

$$= \frac{1}{2} \int_{0}^{\pi^2 / 4} \cos u \, du = \left. \frac{\sin u}{2} \right|_{0}^{\pi^2 / 4} = \frac{1}{2} \sin \left( \frac{\pi^2}{4} \right),$$
\[
\int_0^{\pi/2} \int_0^{x^2} x \sin y \, dy \, dx = \frac{\pi^2}{8} - \frac{1}{2} \sin \left( \frac{\pi^2}{4} \right).
\]

**Problem 5.** Use a double integral in polar coordinates to find the volume of the solid enclosed by the paraboloid \( z = 4 - x^2 - y^2 \) and the plane \( z = 0 \).

**Solution.** The volume of a solid enclosed between the region \( R \) in the plane \( z = 0 \) and the graph of a function \( z = f(r, \theta) \) given in polar coordinates is \( V = \int \int_R f(r, \theta) \, dA \). In our case, the region \( R \) is the intersection of the paraboloid given in rectangular coordinates by the equation \( z = 4 - x^2 - y^2 \) and the plane \( z = 0 \), is the disk \( x^2 + y^2 \leq 4 \). This disk \( R \) is polar coordinates is described as \( 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2 \).

The equation of the paraboloid in polar coordinates is \( z = 4 - r^2 \) (we use the relation \( r^2 = x^2 + y^2 \)). Therefore,

\[
V = \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left[ 2r^2 - \frac{r^4}{4} \right]_{r=0}^{r=2} \, d\theta = \int_0^{2\pi} 4 \, d\theta = 4\theta \bigg|_0^{2\pi} = 8\pi.
\]

**Problem 6.** Express \( \int_0^1 \int_{1-y}^{1-y^2} f(x, y) \, dx \, dy \) as an equivalent repeated integral with the order of integration reversed. (No evaluation of integral is required since no specific expression for \( f(x, y) \) is given. You should sketch the region of integration.)

**Solution.** the region is enclosed between the straight line \( x = 1 - y \) and the parabola \( x = 1 - y^2 \) inside the strip \( 0 \leq y \leq 1 \), where the parabola is to the right (and above) of the straight line (make a picture!). Solving for \( y \), we obtain the equations \( y = 1 - x \) for the straight line, and \( y = \sqrt{1 - x} \) for the parabola. Since at the intersection of the two lines we have \( 1 - x = \sqrt{1 - x} \), either \( x = 0 \) or \( x = 1 \). Thus, we obtain that the region of integration is given by the inequalities \( 0 \leq x \leq 1, 1 - x \leq y \leq \sqrt{1 - x} \). Therefore,

\[
\int_0^1 \int_{1-y}^{1-y^2} f(x, y) \, dx \, dy = \int_0^1 \int_{1-x}^{\sqrt{1-x}} f(x, y) \, dy \, dx.
\]

**Problem 7.** A unit ball \( x^2 + y^2 + z^2 \leq 1 \) is made from an inhomogeneous material whose density is \( \rho(x, y, z) = 2 - z^2 + x^2 + y^2 \). Find the maximum and the minimum of density within the ball.
Solution. We will first find the critical points of the function $\rho(x, y, z)$ in the interior of the ball, $x^2 + y^2 + z^2 < 1$, solving the equations

$$
\rho_x(x, y, z) = 2x = 0, \quad \rho_y(x, y, z) = 2y = 0, \quad \rho_z(x, y, z) = -2z = 0.
$$

They have a unique solution $x = 0, y = 0, z = 0$, i.e., there is only one critical point, $P(0, 0, 0)$ inside the ball. It is also clear that on the boundary $x^2 + y^2 + z^2 = 1$ we have

$$
\rho(x, y, z) = 2 - z^2 + (1 - z^2) = 3 - 2z^2.
$$

Since $0 \leq z^2 \leq 1$, at the boundary points where an extremum can occur we have $z = 0$ (and correspondingly, any $x$ and $y$ with $x^2 + y^2 = 1$) or $z = \pm 1$ (and correspondingly, $x = y = 0$). At the points $(x, y, 0)$ with $x^2 + y^2 = 1$ we have $\rho(x, y, 0) = 3 - 2 \cdot 0^2 = 3$. At the points $(0, 0, \pm 1)$ we have $\rho(0, 0, \pm 1) = 3 - 2 \cdot 1 = 1$. Finally, at the critical point $P(0, 0, 0)$ which is inside the ball we have $\rho(0, 0, 0) = 2 - 0 + 0 + 0 = 2$. Therefore, the maximal value of $\rho(x, y, z)$ is 3 and the minimal value is 1. Both these values are attained at the boundary of the ball.